

Push-Relabel Max Flow

"Average" analysis of algorithms

↑
(without sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

0, 1, 10, 11, 100, 101, ...
1111111111111111
 1000000000000000

n increments $\Rightarrow O(n)$ total time
each increment takes $O(1)$ on average

" $O(1)$ amortized time"

- 1st* bit

- 2nd bit

- 3rd bit

4

- kth bit

how often total

every time n

every 2 times $\lfloor \frac{n}{2} \rfloor$

every 4 times $\lfloor \frac{n}{4} \rfloor$

every 2^{k-1} times $\lfloor \frac{n}{2^{k-1}} \rfloor$

Amortized Analysis

"T amortized time"

$\Rightarrow$ n operations take $O(nT)$ time

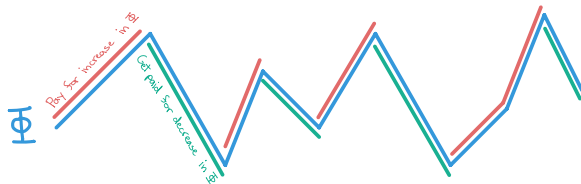
Multiple operations $OP_1, \dots, OP_K$

"OP₁ takes T_1 amortized time, OP₂ take $\approx$

OP_k takes T_k amortized time"

Potential method

Invent "potential Φ "



Amortized cost of operation will be

$$(\text{real time}) + \Delta\Phi$$

Credit schemes Idea: buy credits ahead of time to pay for work later

(running time) + $\alpha(1) \cdot (\Delta \text{ credit}) \leq \text{amortized time}$ put: 1 credit on first bit
(not increase) 1 credit on each bit

e.g. for binary counter, when incrementing:

when we need to flip bit i , we have already saved 1 credit to pay for it

put: 1 credit on first bit

$\frac{1}{2}$ credit on 2nd bit

$\frac{1}{4}$ credit on 3rd bit

+ $\frac{1}{2^{i-1}}$ credits on i th bit

2 credits total

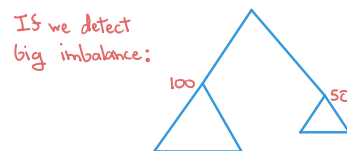
e.g. binary counter

$\Phi = \# \text{ bits set to } 1$

then for each inc,

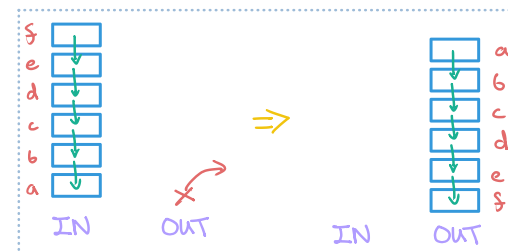
$$(\text{real time}) + \Delta \bar{\Phi} \leq O(1)$$

leach bit reset to 0 paid by $\Delta\Phi$


$$\left(\begin{array}{c} \# \text{ in one} \\ \text{subtree} \end{array} \right) > 2 \times \left(\begin{array}{c} \# \text{ in other} \\ \text{subtree} \end{array} \right)$$

rebuild entire subtree tree optimally
(choose median to be root of subtree, recurse) (linear time)

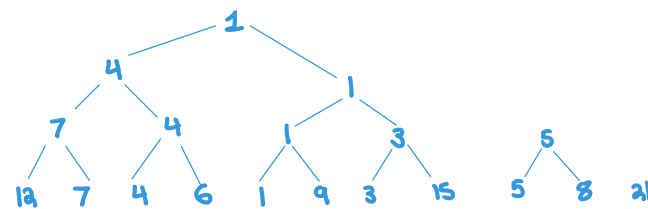
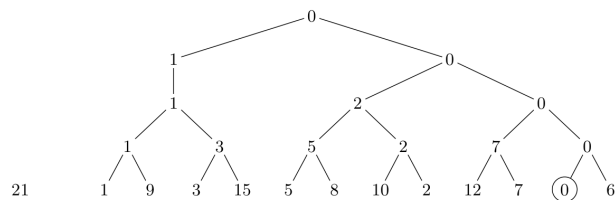
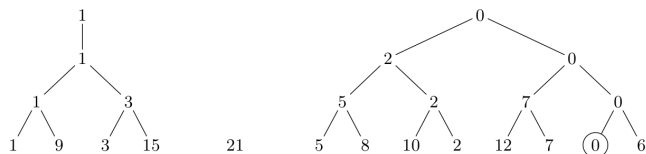
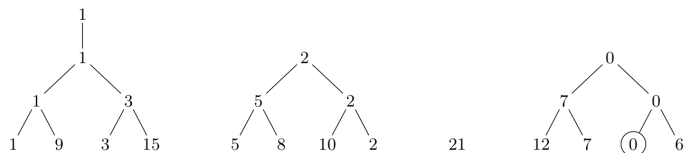
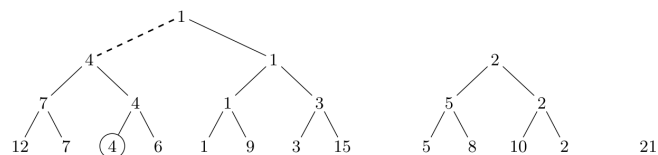
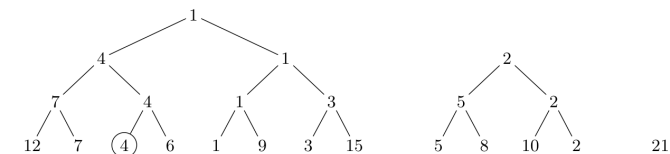
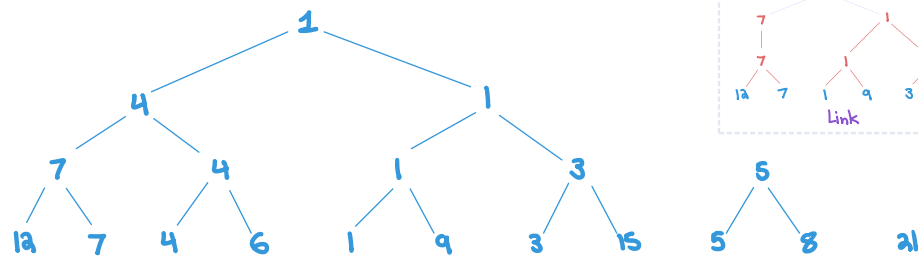
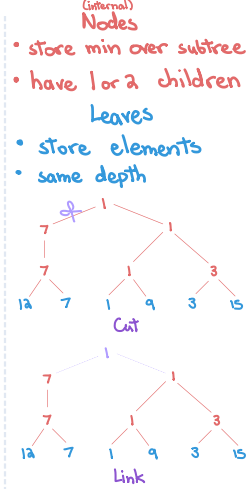
Doubling Trick



"Heap" = Collection of tournament trees:

Rules:

- ① All tree heights distinct
- ② Max height $O(\log n)$



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Maintaining

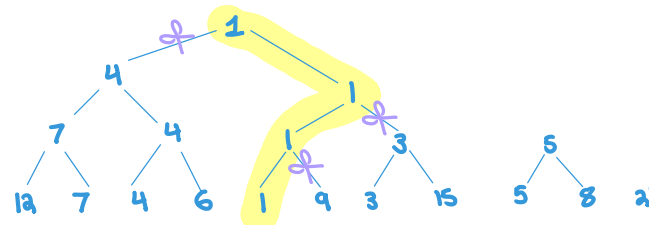
$$(\# \text{ unary nodes at height } h) \geq \frac{1}{2} (\# \text{ nodes at height } h)$$

$\Rightarrow$ delete all nodes w/ height $\geq h$

$$\Phi = C_t (\# \text{ trees}) + C_u (\# \text{ unary nodes}) + C_n (\# \text{ nodes})$$

pays for link pays for new trees if $C_u > 2C_t$ pays for deletion

$$\Phi = C_t (\# \text{ trees}) + C_u (\# \text{ unary nodes}) + C_n (\# \text{ nodes})$$



- ① All tree heights distinct
 - ② Max height $O(\log n)$
- } paid by Φ

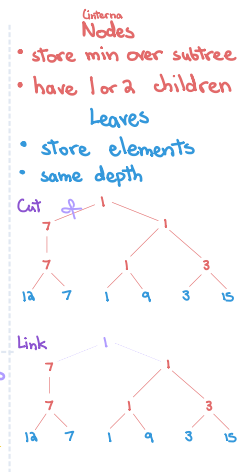
3. remove-min(): removes and returns the key value pair (k, p) with the minimum priority p .

1. find min by scanning trees

2. cut all subtrees

(and cleanup)

$O(\log n)$ work (excluding cleanup)
+ $O(\log n)$ increase in Φ
 $\underline{O(\log n) \text{ amortized time}}$



today: push relabel algo. for max flow

- fastest algos we will see for flow

why?

(w/r/t class topics)

Roadmap for rest of semester

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(w/out sacrificing worst-case robustness)

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uses amortized analysis very nicely

(gives us an example that is not a data structure)

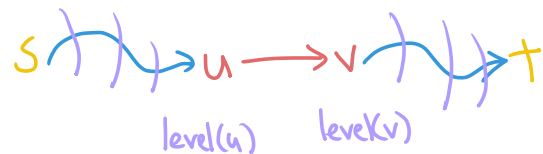
Analysis

- ① while there is a path $p: s \rightsquigarrow t$:
 - Ⓐ take shortest path p and route as much flow as possible
 - Ⓑ update residual graph

Key
Lemma

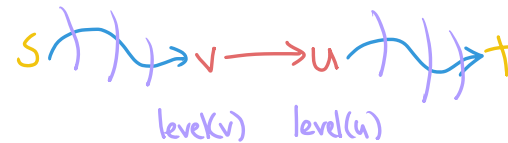
$\forall v \in V$, distance from s to v in residual graph never decreases

① e removed from ^(residual graph) RG,
 $\Rightarrow$ e in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(v) = \text{level}(u) + 1$$

② e = (u $\rightarrow$ v) added back in to RG
 $\Rightarrow$ (v $\rightarrow$ u) was in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(u) = \text{level}(v) + 1$$

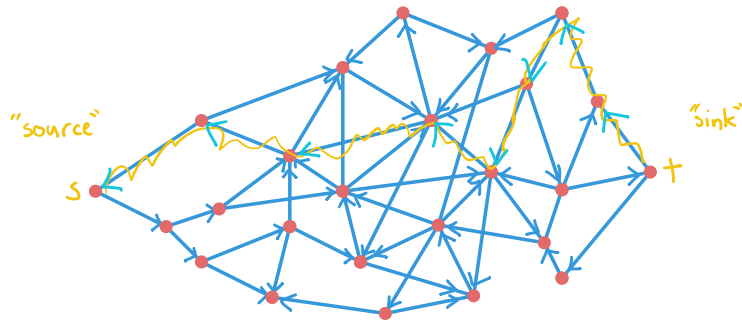
each event ①/② $\Rightarrow$ level(u), level(v) have switched order.

level's nondecreasing by key lemma

$\Rightarrow$ e gets deleted/added at most n times

emphasis on distances in residual graph

Edmonds-Karp



- ① while there is a path $p: s \rightarrow t$:
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Analysis

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Key Lemma $\forall v \in V$, distance from s to v in residual graph never decreases

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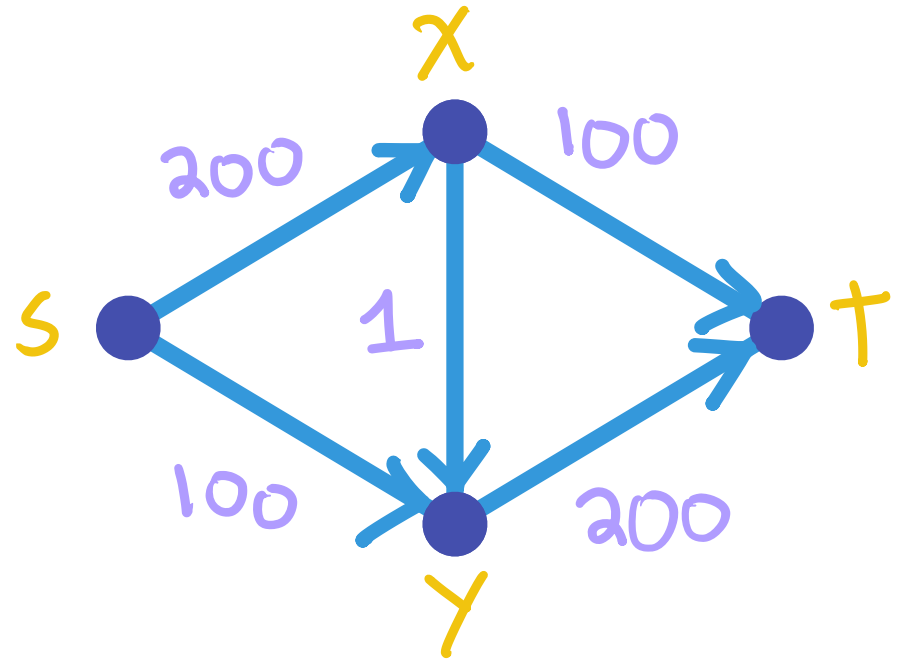
emphasis on distances in residual graph

Push-Relabel

"push flow" one edge
at a time

Which "push"?

How to measure progress?



Framework

① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$

↪ Relaxation of flows, allows intermediate excess

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

↪ Relaxation of distances in residual graph

Push-Relabel Framework

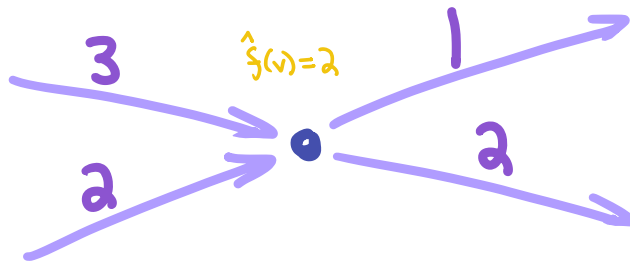
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↖ Relaxation of flows

- capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(e) - \sum_{e \in \mathcal{I}^+(v)} f(e)$$



net flow at v

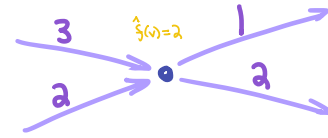
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↖ Relaxation of distances in residual graph

Push-Relabel Framework

Residual graph $G_f = (V, E_f)$

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Relaxation of flows



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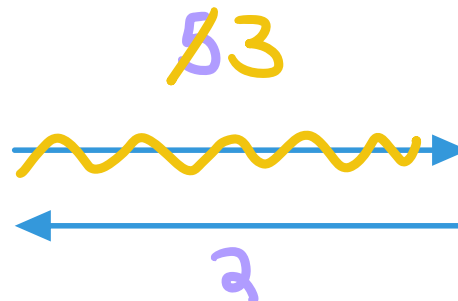
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Relaxation of distances in residual graph

just like for flows:

$$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$$

residual capacity



② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\uparrow$ Relaxation of distances in residual graph

• Sink has level 0, source has level n

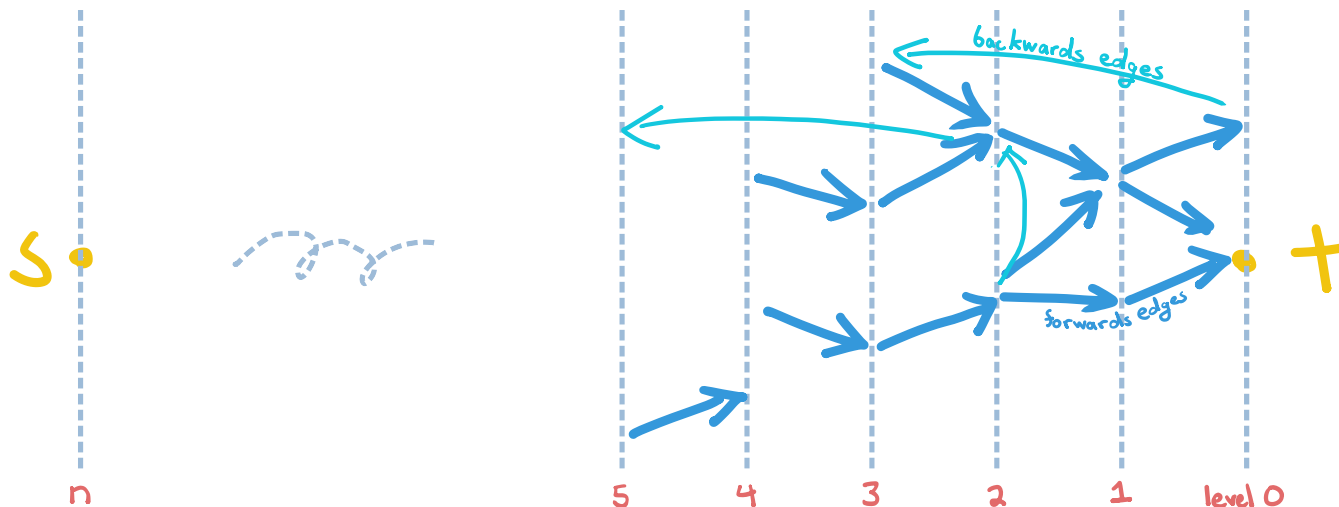
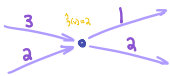
$$l(t) = 0, \quad l(s) = n$$

• edges (in resid. graph) go down ≤ 1 level

$$l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$$

Push-Relabel Framework

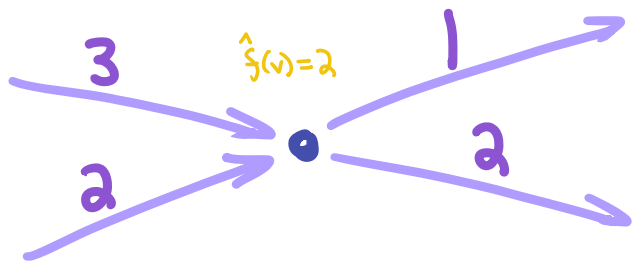
- ① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
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 $\underbrace{\hspace{10em}}_{\text{net flow at } v}$



Push-Relabel Framework

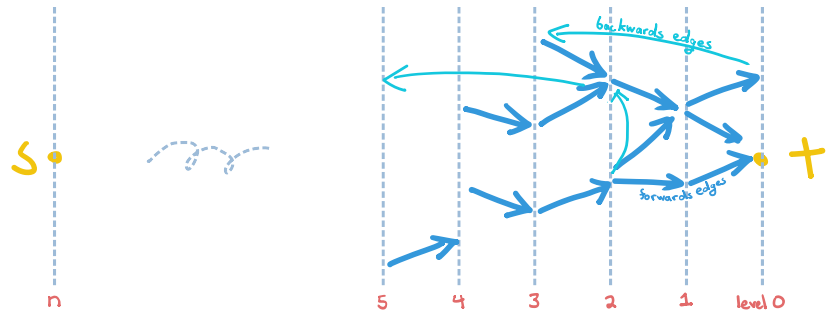
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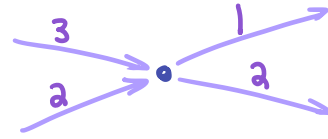
Initially: saturate all source edges, set all non-terminal levels to 0

$$f(s,v) = c(s,v) \quad \forall (s,v) \in \mathcal{I}^+(s)$$

$$\text{level}(s) = n, \quad \text{level}(v) = 0 \quad \forall v \neq t$$

call a vertex u "active" if

$$\hat{f}(u) > 0$$



high-level goal:

push flow out of u "towards t "

Push-Relabel Operations

① Push flow along forward edge

② Relabel vertex to higher level

(while following the rules)

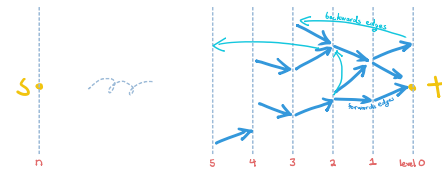
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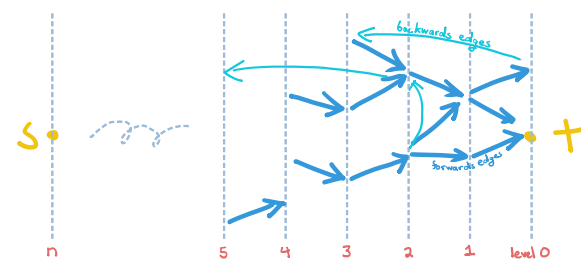
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① Push flow along forward edge

$$e = (u, v) \in E_f \text{ w/ } l(u) = l(v) + 1$$

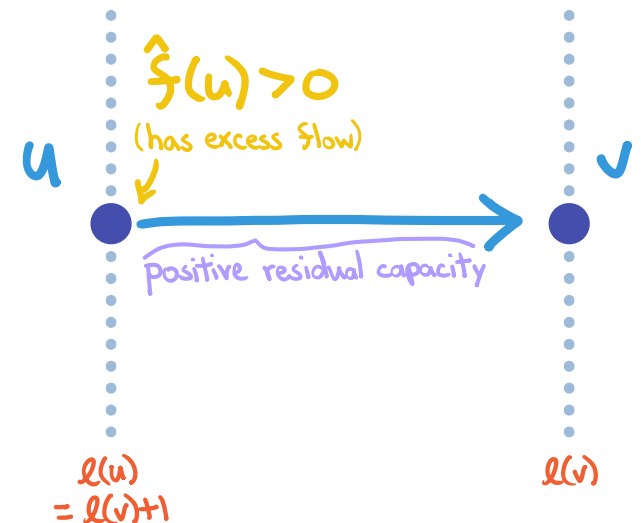
u active

(usually)

push as much flow as possible:

$$\min \{ c_f(e), \hat{f}(u) \}$$

(residual capacity) (excess flow)



② Relabel vertex to higher level

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
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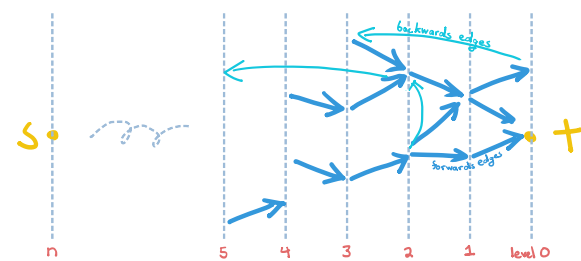
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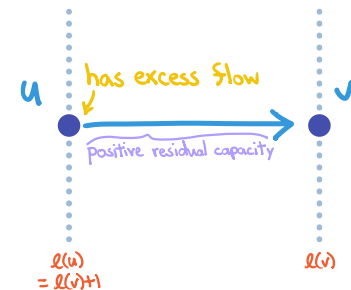
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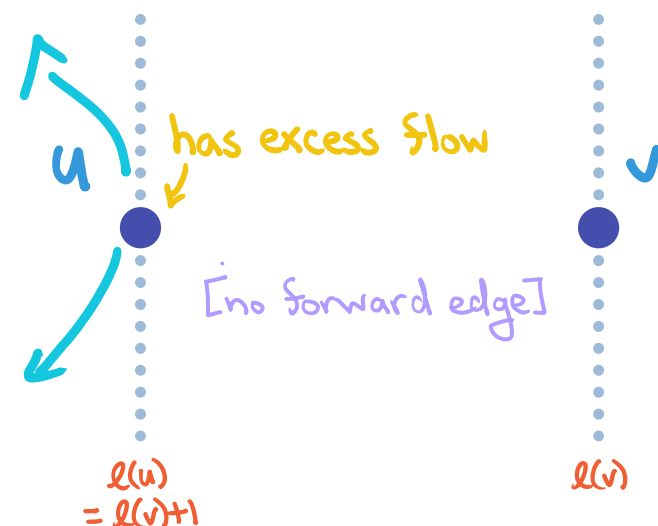
① Push flow along forward edge



② Relabel vertex to higher level

u active, no forward edge

safe to set $l(u) = l(u) + 1$



Algorithm:

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

WTF:

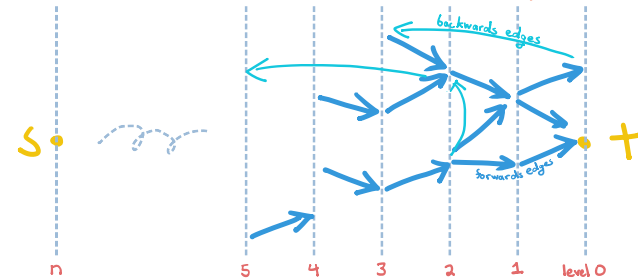
- flow? (let alone max flow?)
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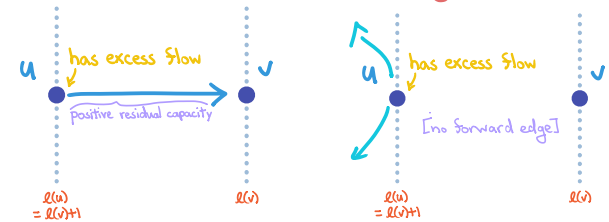
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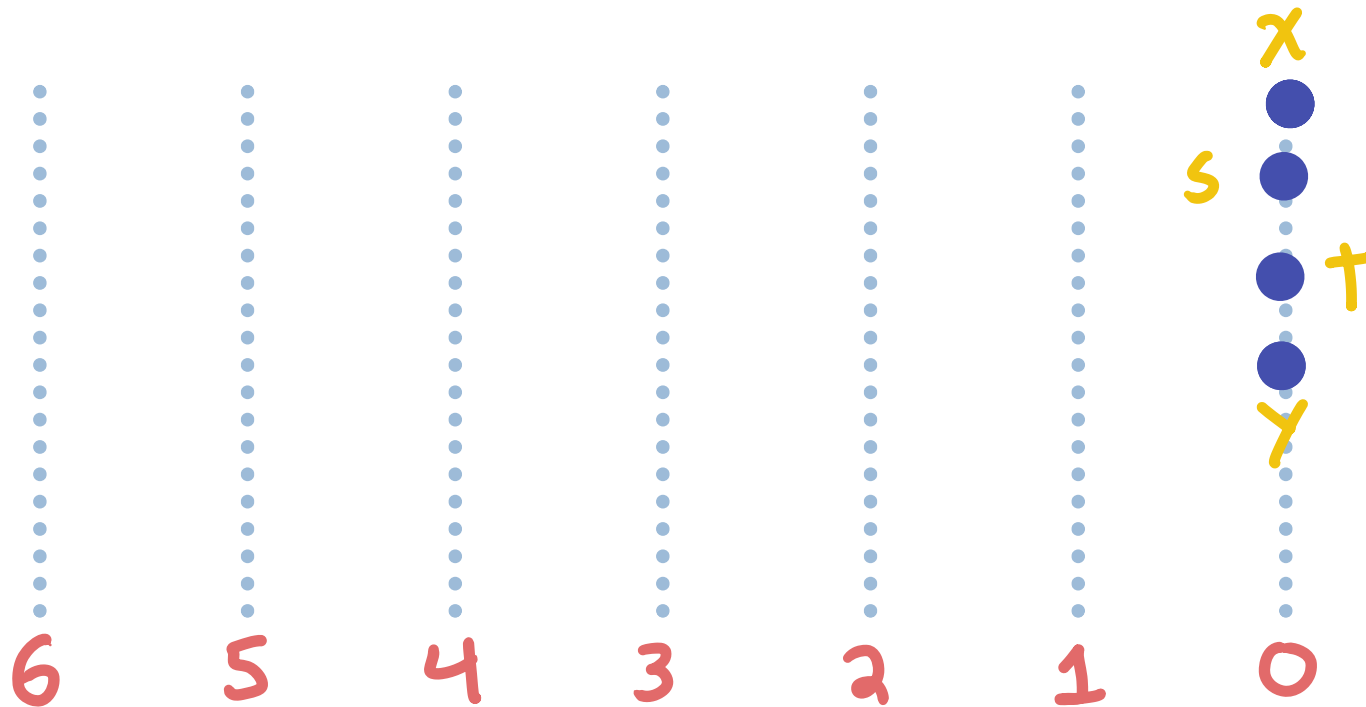
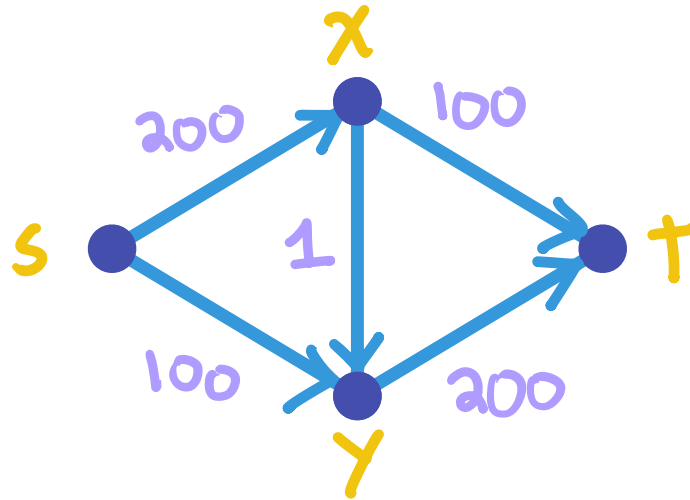
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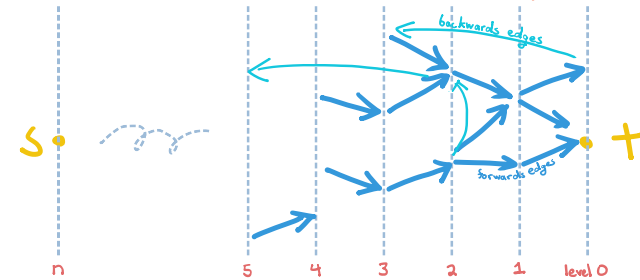
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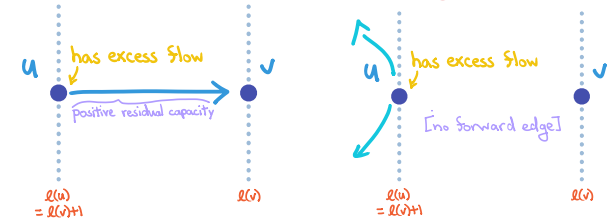
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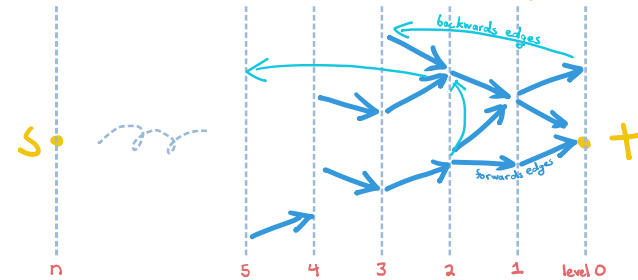
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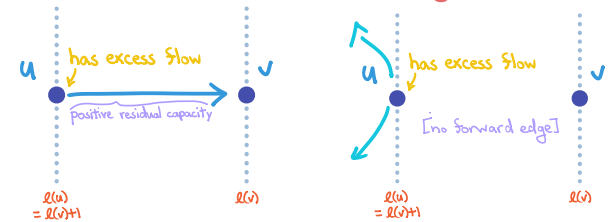
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- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Understanding "preflows"

Lemma let f be a preflow.
then f is a path packing w/
 $\hat{f}(v)$ (s,v) -paths for all v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(u,v) - \sum_{e \in \mathcal{I}^+(v)} f(v,u)$

Understanding "labels"

Lemma $\forall x, y \in V,$

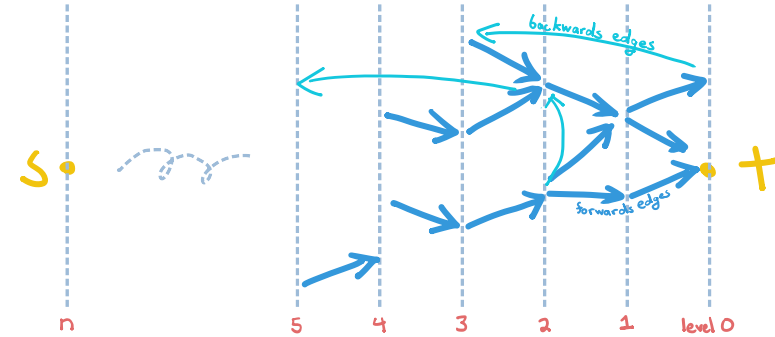
$$l(x) - l(y) \leq d(x, y)$$

where $d(x, y)$ = distance in resid. graph

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

- Sink has level 0, source has level n
 $l(t) = 0, l(s) = n$
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Understanding "labels"

Lemma $\forall x, y \in V,$
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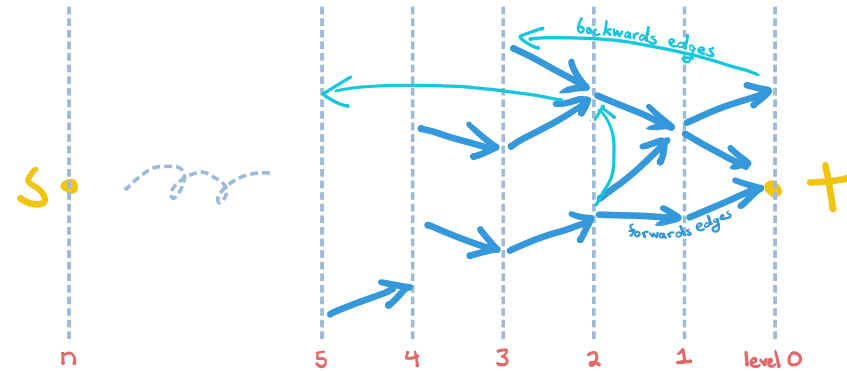
$$\Rightarrow d(s, t) \geq l(s) - l(t) = n$$

Corollary: No (s, t) -path in residual graph!

(always an (s, t) -cut in residual graph
separated by empty layers)

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- Sink has level 0, source has level n
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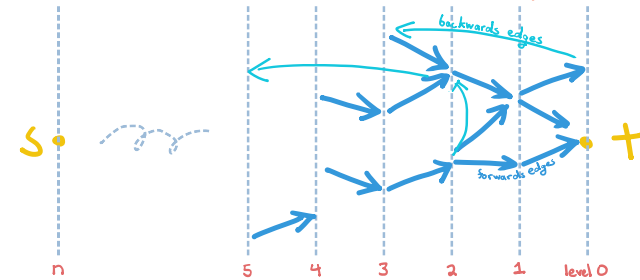
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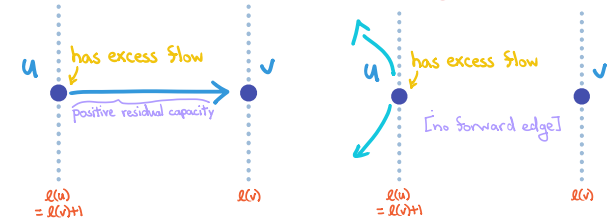
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
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① Push flow along forward edge

② Relabel vertex to higher level



Lemma if v active, then
 $l(v) < 2n$

Lemma f preflows, $l: V \rightarrow \mathbb{Z}_{\geq 0}$ levels

if f is a flow, then

① f is a max flow.

② there is empty layer $i \in \{1, \dots, n-1\}$

s.t. for $U = \{v: l(v) > i\}$,

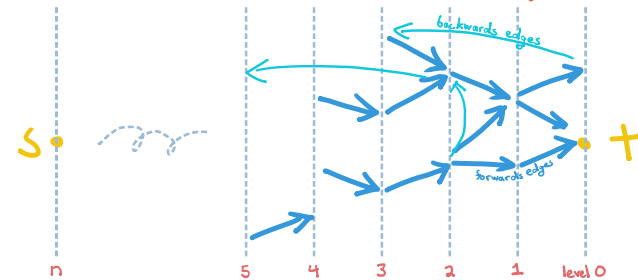
$\partial^+(U)$ is a min (s, t) -cut

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \partial^-(v)} f(e) - \sum_{e \in \partial^+(v)} f(e)$

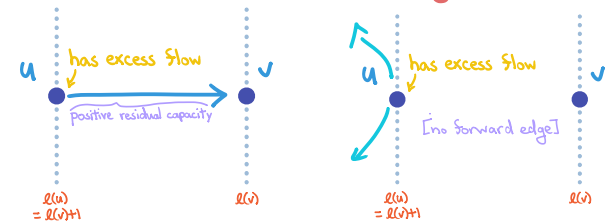
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
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① Push flow along forward edge

② Relabel vertex to higher level



Lemma f preflows, $l: V \rightarrow \mathbb{Z}_{\geq 0}$ levels

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$\partial^+(U)$ is a min (s,t) -cut

(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

when no v 's are active,
 f is a max flow.

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

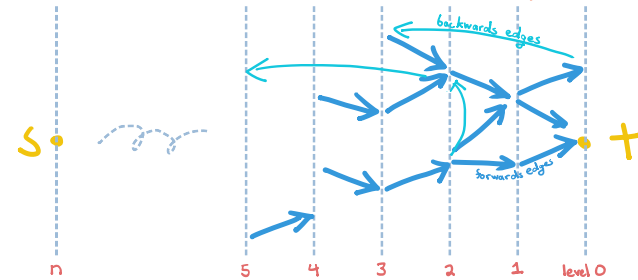
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

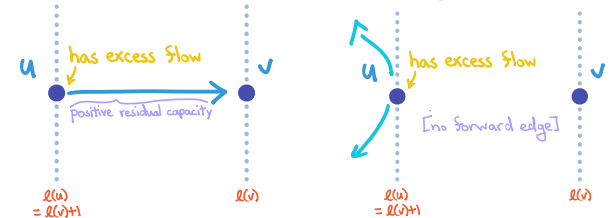
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(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

Now we know why algo is correct.

next: running time

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

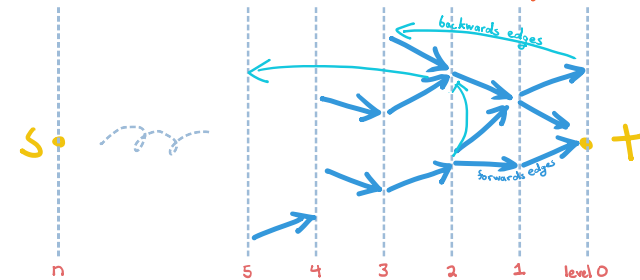
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

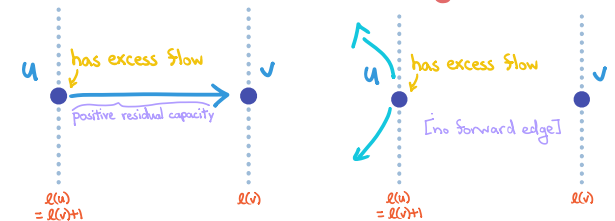
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① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

Two kinds of pushes:

① Saturating push

uses all capacity of edge

② Non-saturating push

instead pushes out all excess flow from v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

• capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

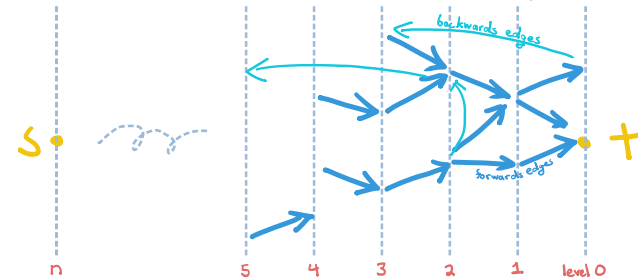
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of distances in residual graph

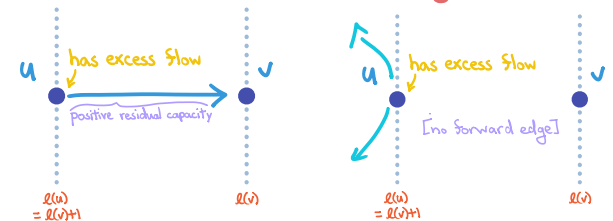
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① Push flow along forward edge

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(Algorithm)

- ① while there is active $v \in V - \{s, t\}$
 - Ⓐ try to push flow out of v
 - Ⓑ otherwise relabel v

we will bound:

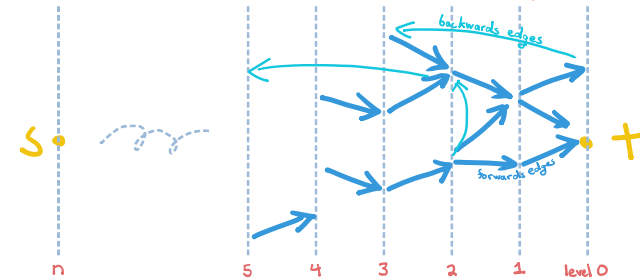
- ① # relabelings
- ② # saturating push
- ③ # non-saturating pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

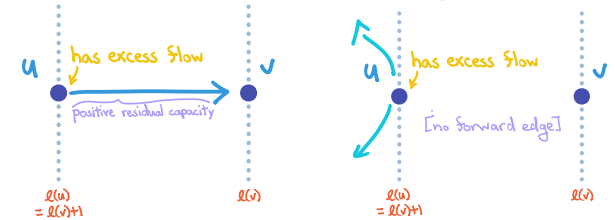
- capacity constraints $0 \leq f(e) \leq c(e) \quad \forall e \in E$
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- ① Push flow along forward edge
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Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

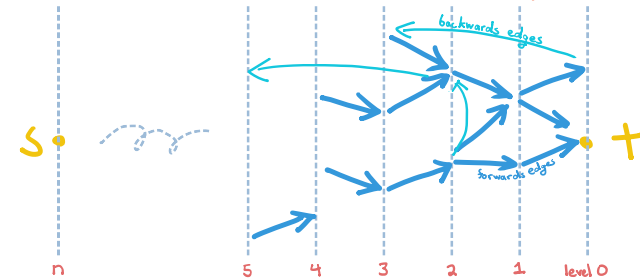
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- ② try to push flow out of v
- ③ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

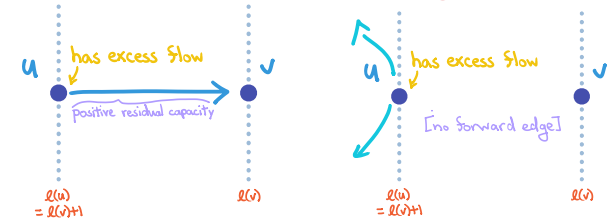
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- ① Push flow along forward edge
- ② Relabel vertex to higher level



Lemma

(# saturating pushes) $\leq O(mn)$

($O(n)$ per edge)

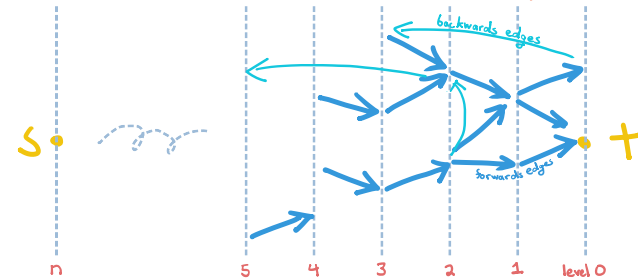
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① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

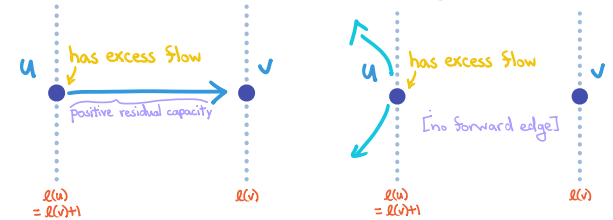
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Lemma

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

Proof $\Phi = \sum_{\text{active } v} l(v)$

Φ nonnegative,
initially 0

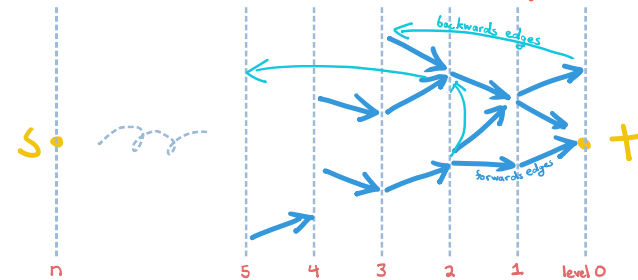
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① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

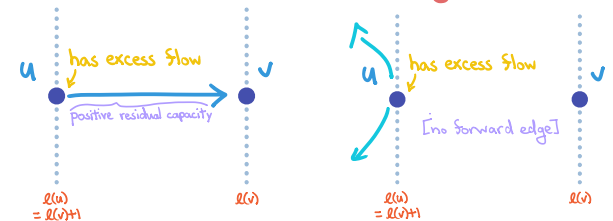
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- ① Push flow along forward edge
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(Algorithm)

① while there is active $v \in V - \{s, t\}$

 Ⓐ try to push flow out of v

 Ⓑ otherwise relabel v

$$(\# \text{ relabels}) \leq O(n^2)$$

$$(\# \text{ saturating pushes}) \leq O(mn)$$

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

+

$$O(mn^2)$$

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
~ Relaxation of flows

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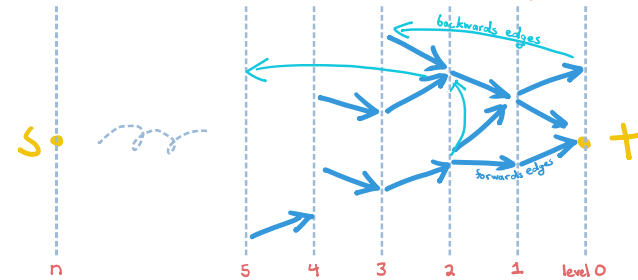
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
~ Relaxation of distances in residual graph

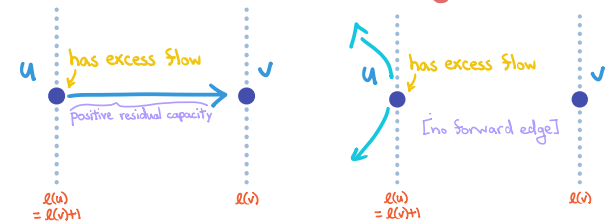
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① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

 Ⓐ try to push flow out of v

 Ⓑ otherwise relabel v

$O(mn^2)$ time

faster?

Bottleneck was non-saturating pushes

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

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• nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

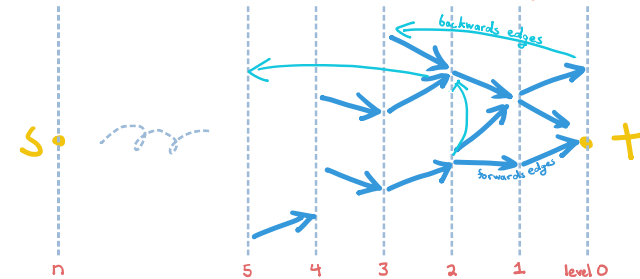
$$\text{where } \hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} \hat{f}(u, v) - \sum_{e \in \mathcal{O}(v)} \hat{f}(v, u)$$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

Relaxation of distances in residual graph

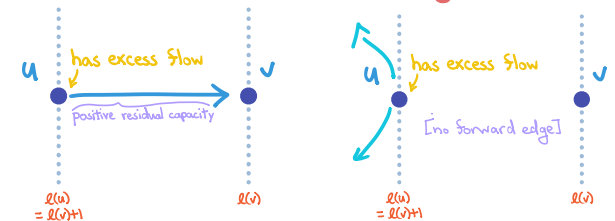
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① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

① while there is active $v \in V - \{s, t\}$

★ let $v =$ active w/ highest level

@ try to push flow out of v

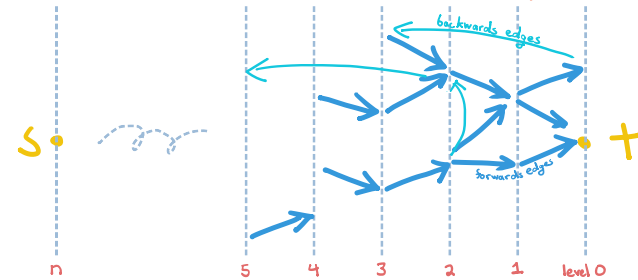
⑥ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
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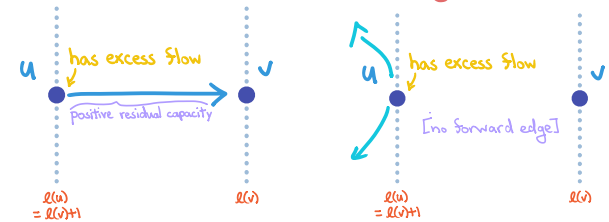
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① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ★ let $v =$ active w/ highest level
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn)$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

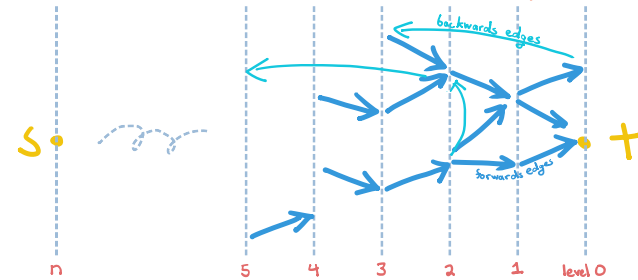
↑ better?

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

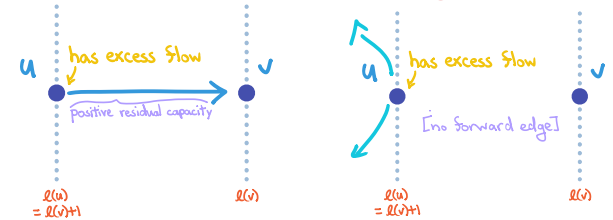
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- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
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- ① Push flow along forward edge
- ② Relabel vertex to higher level

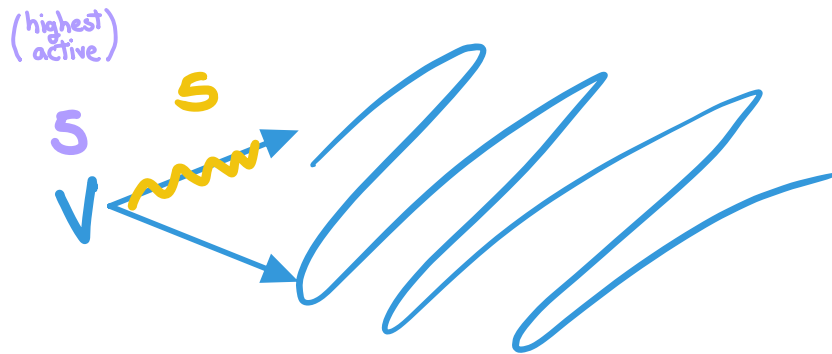


Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$

proof.

suppose non satur. push from v



Top-down push-relabel

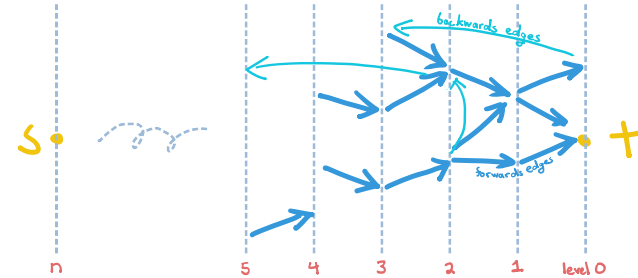
- ① while there is active $v \in V - \{s, t\}$
- ② let $v =$ active w/ highest level
- ③ try to push flow out of v
- ④ otherwise relabel v

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

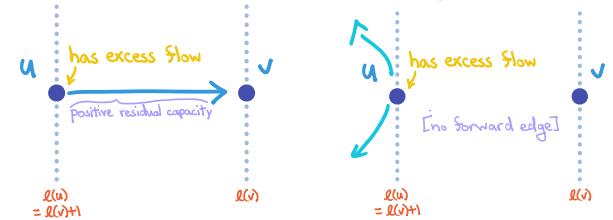
- capacity constraints
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$
- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$
 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



- ① Push flow along forward edge
- ② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ★ let $v =$ active w/ highest level
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 - ⓑ otherwise relabel v

Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn)$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$

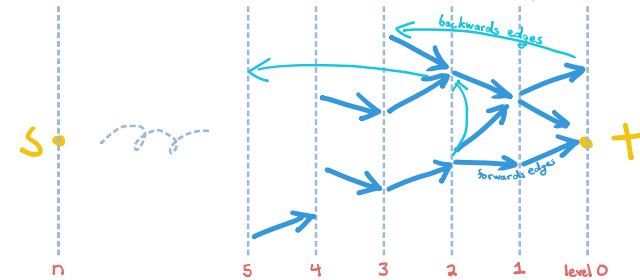
$\Rightarrow O(n^3)$ time overall.

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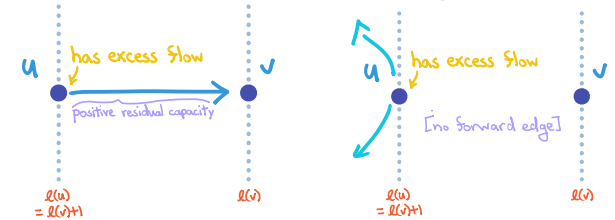
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(better?)

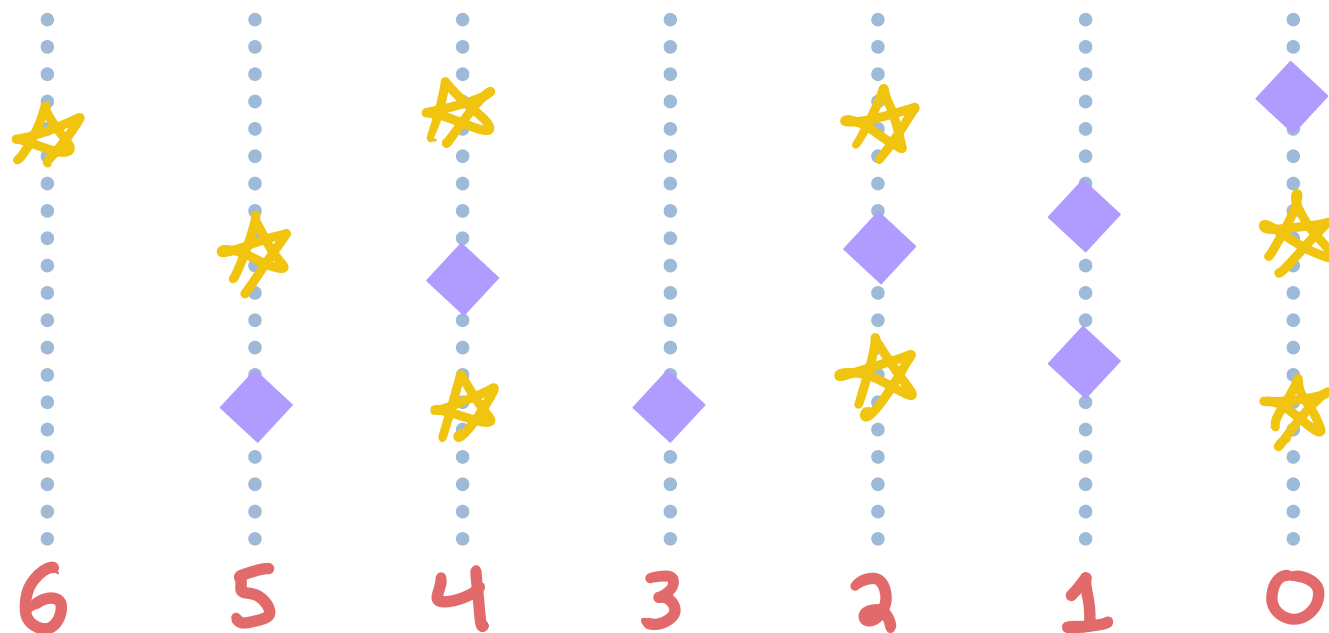
Lemma (# non-satur. pushes) $\leq O(n^2 \sqrt{m})$

proof.

$$\Phi = \sum_{v \text{ active}} |w: l(w) \leq l(v)|$$

★ = active

◆ = inactive
15 total

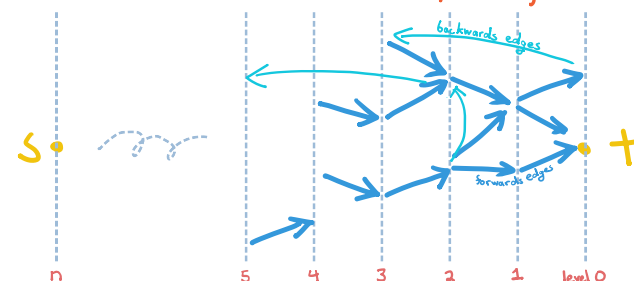


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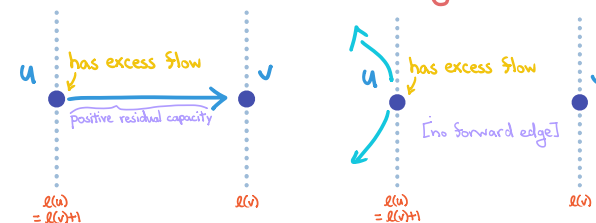
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proof. $\Phi = \sum_{v \text{ active}} |w: l(w) \leq l(v)|$

relabel, satur. push: $\Delta \Phi \text{ by } \leq n$
 $\Rightarrow$ total increase of $O(mn^2)$

non-satur. push at v : $\Delta \Phi = -|w: l(w) = l(v)|$

"big" if $|w: l(w) = l(v)| \geq k$ w/ $k \text{ TBD}$

"small" if $|w: l(w) = l(v)| \leq k$

claims: $O(mn^2/k)$ big pushes

$O(n^2 k)$ small pushes

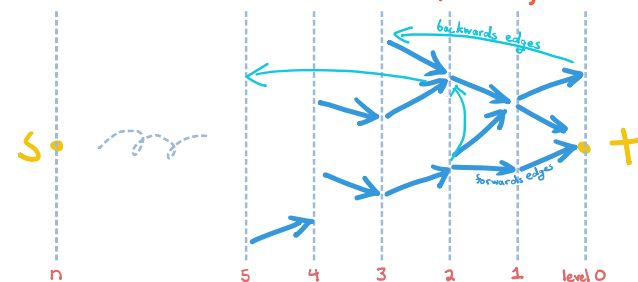
Then set $k = \sqrt{m}$

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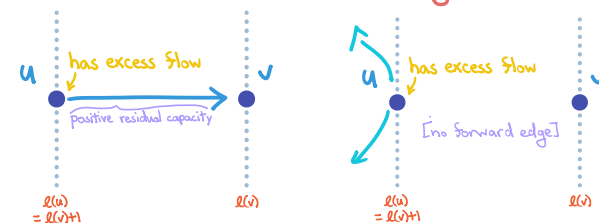
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claim: $O(mn^2/k)$ big pushes

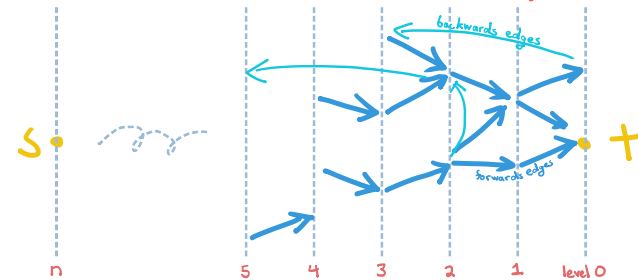
big push decreases Φ by $\geq k$

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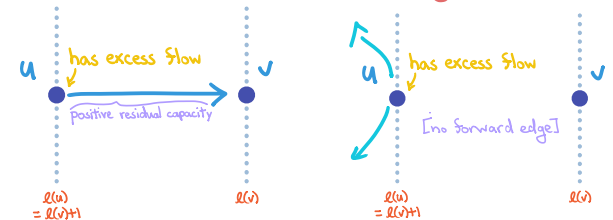
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claim: $O(n^2 k)$ small pushes why?

k "small" pushes before active level decreases

n^2 active level decreases

$\hookrightarrow n^2$ active level increases

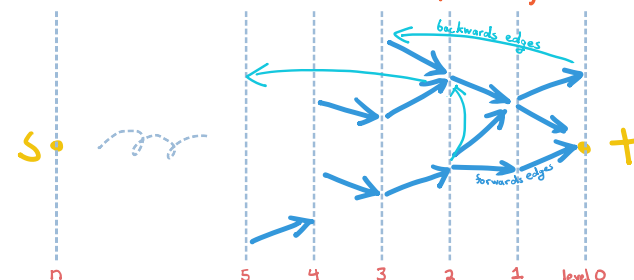
because n^2 relabels

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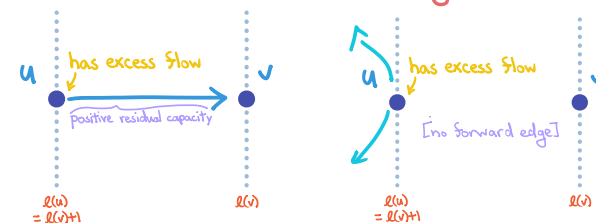
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