

Push-Relabel Max Flow

"Average" analysis of algorithms

(without sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

0, 1, 10, 11, 100, 101, ...
1111111111111111
 1000000000000000

n increments $\Rightarrow O(n)$ total time
each increment takes $O(1)$ on average

" $O(1)$ amortized time"

- 1st* bit

- 2nd bit

- 3rd bit

4

- kth bit

how often total

every time n

every 2 times $\lfloor \frac{n}{2} \rfloor$

every 4 times $\lfloor \frac{n}{4} \rfloor$

every 2^{k-1} times $\lfloor \frac{n}{2^{k-1}} \rfloor$

Amortized Analysis

"T amortized time"

$\Rightarrow$ n operations take $O(nT)$ time

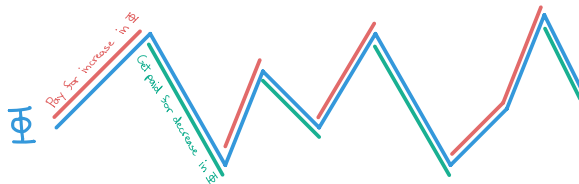
Multiple operations $OP_1, \dots, OP_K$

"OP₁ takes T_1 amortized time, OP₂ take $\approx$

OP_k takes T_k amortized time"

Potential method

Invent "potential Φ "



Amortized cost of operation will be

$$(\text{real time}) + \Delta\Phi$$

Credit schemes Idea: buy credits ahead of time to pay for work later

$$(\text{running time}) + \underbrace{O(1) \cdot (\Delta \text{ credit})}_{\text{let increase}} \leq \text{amortized time} \quad \text{put: } \begin{matrix} 1 & \text{credit on first bit} \\ \text{bit} & \text{bit} \end{matrix}$$

e.g. for binary counter, when incrementing:

when we need to flip bit i , we have already saved 1 credit to pay for it

put: 1 credit on first bit

$\frac{1}{2}$ credit on 2nd bit

$\frac{1}{4}$ credit on 3rd bit

+ $\frac{1}{2^{i-1}}$ credits on i th bit

2 credits total

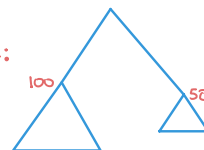
e.g. binary counter

$\Phi = \# \text{ bits set to } 1$

then for each inc,

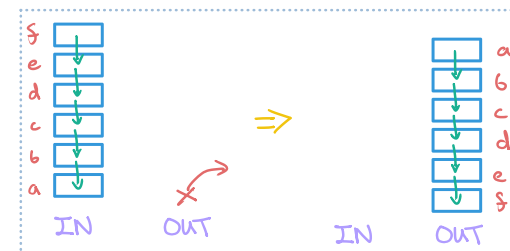
$$(\text{real time}) + \Delta \bar{\Phi} \leq O(1)$$

leach bit reset to 0 paid by $\Delta\Phi$


$$\left(\begin{array}{c} \# \text{ in one} \\ \text{subtree} \end{array} \right) > 2 \times \left(\begin{array}{c} \# \text{ in other} \\ \text{subtree} \end{array} \right)$$

rebuild entire subtree tree optimally
(choose median to be root of subtree, recurse) (linear time)

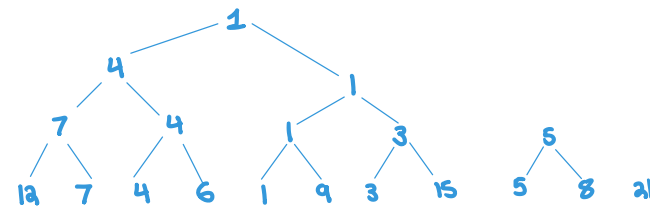
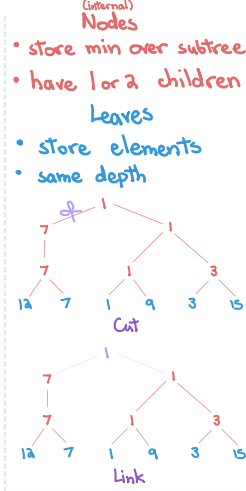
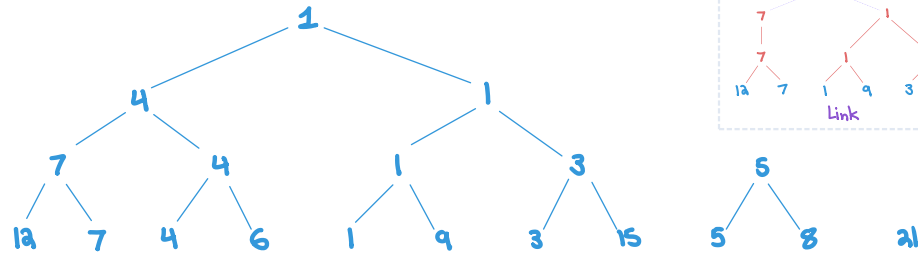
Doubling Trick



"Heap" = Collection of tournament trees:

Rules:

- ① All tree heights distinct
- ② Max height $O(\log n)$



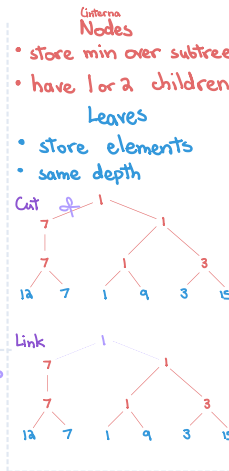
- Rules:
- ① All tree heights distinct
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Maintaining

$$(\# \text{ unary nodes at height } h) \geq \frac{1}{2} (\# \text{ nodes at height } h)$$

$\Rightarrow$ delete all nodes w/ height $\geq h$

Risk: many unary nodes



$$\Phi = C_t (\# \text{ trees}) + C_u (\# \text{ unary nodes}) + C_n (\# \text{ nodes})$$

constant TBO

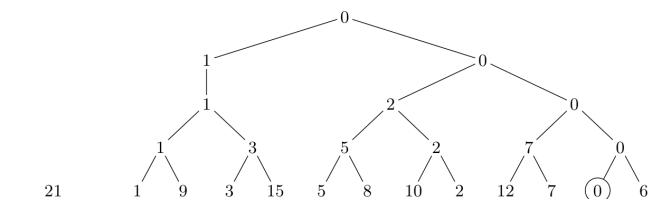
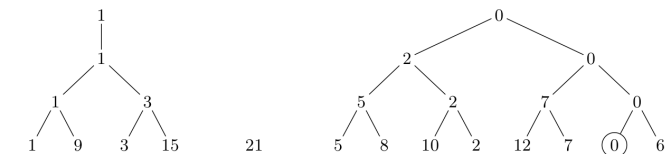
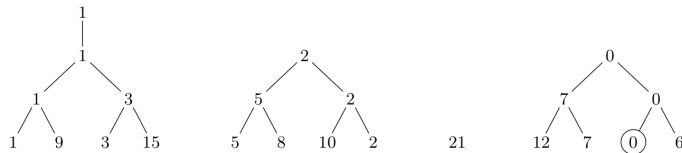
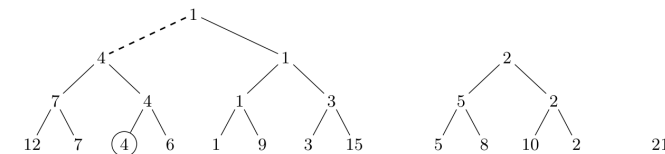
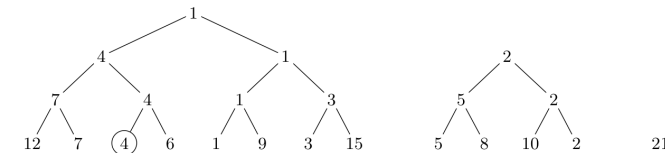
pays for link

constant TBO

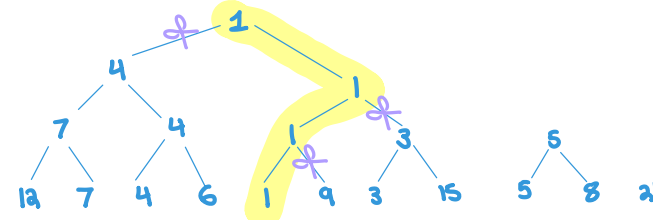
pays for new trees if $C_u > 2C_t$

constant TBO

pays for deletion



$$\Phi = C_t (\# \text{ trees}) + C_u (\# \text{ unary nodes}) + C_n (\# \text{ nodes})$$



- ① All tree heights distinct
 - ② Max height $O(\log n)$
- } paid by Φ

3. remove-min(): removes and returns the key value pair (k, p) with the minimum priority p .

1. find min by scanning trees

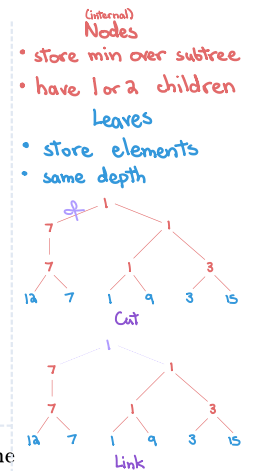
2. cut all subtrees

(and cleanup)

$O(\log n)$ work (excluding cleanup)

+ $O(\log n)$ increase in Φ

$O(\log n)$ amortized time



today: push relabel algo. for max flow

- fastest algos we will see for flow

why?

(w/r/t class topics)

Roadmap for rest of semester

"Average" analysis of algorithms
↑
(w/out sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

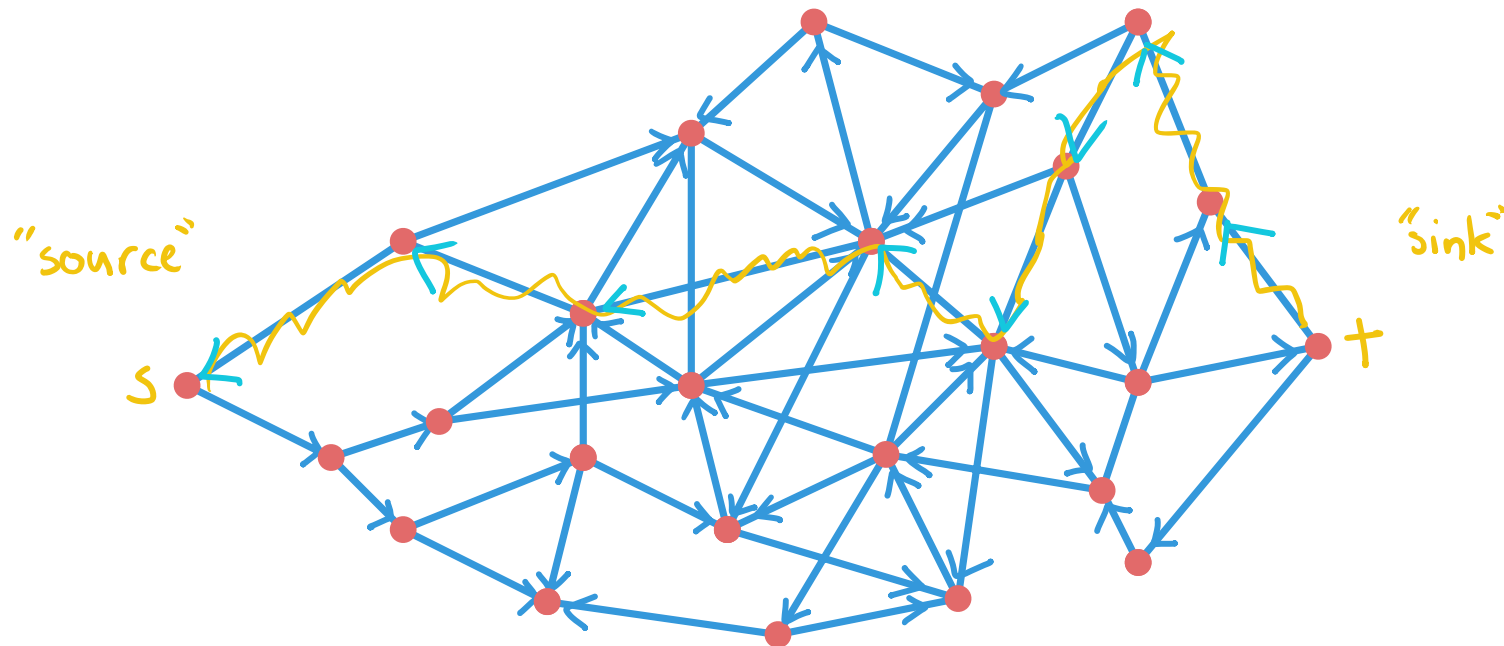
② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

uses amortized analysis very nicely

(gives us an example that is not a data structure)

Edmonds-Karp



① while there is a path $p: s \rightsquigarrow t$:

 Ⓐ take shortest path p and route as much flow as possible

 Ⓑ update residual graph

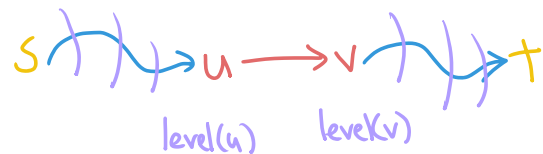
Analysis

- ① while there is a path $p: s \rightsquigarrow t$:
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Key
Lemma

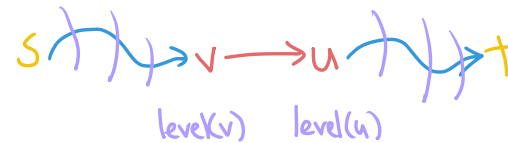
$\forall v \in V$, distance from s to v in residual graph never decreases

① e removed from ^(residual graph) RG,
 $\Rightarrow$ e in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(v) = \text{level}(u) + 1$$

② e = (u $\rightarrow$ v) added back in to RG
 $\Rightarrow$ (v $\rightarrow$ u) was in shortest $s \rightsquigarrow t$ path



$$\Rightarrow \text{level}(u) = \text{level}(v) + 1$$

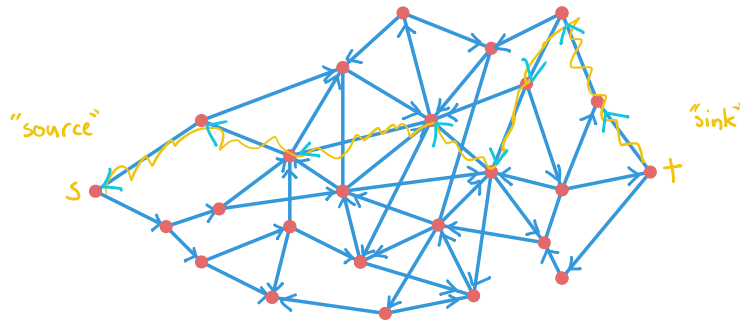
each event ①/② $\Rightarrow$ level(u), level(v) have switched order.

level's nondecreasing by key lemma

$\Rightarrow$ e gets deleted/added at most n times

emphasis on distances in residual graph

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Analysis

- ① while there is a path $p: s \rightarrow t$:
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Key Lemma $\forall v \in V$, distance from s to v in residual graph never decreases

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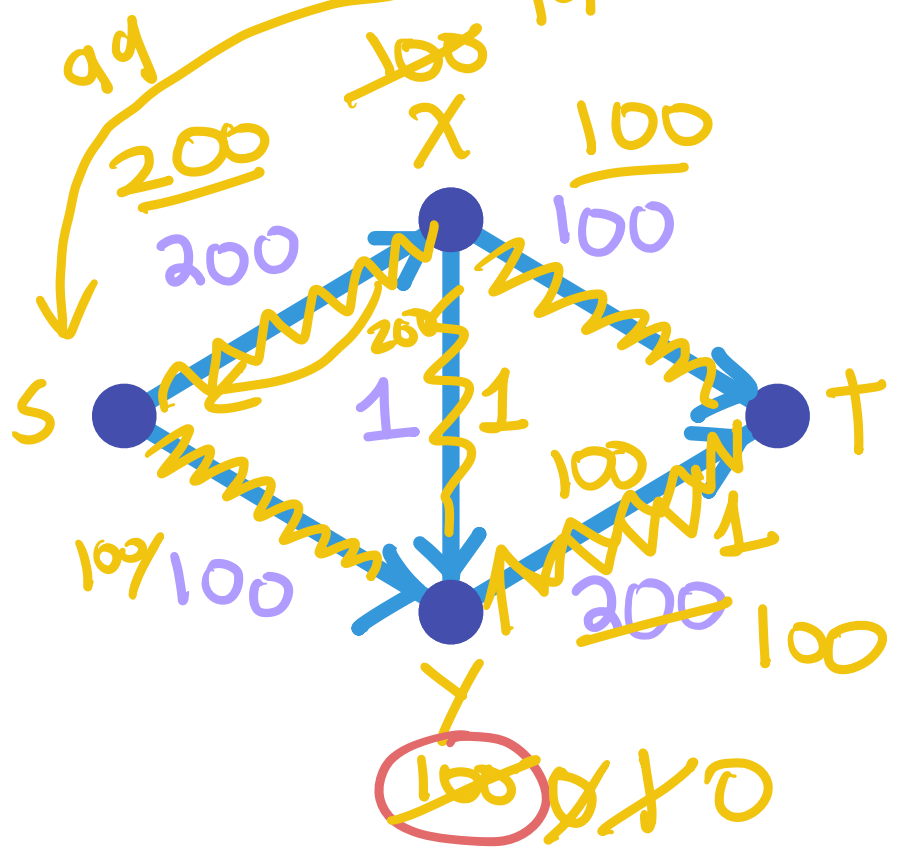
emphasis on distances in residual graph

Push-Relabel

"push flow" one edge at a time

Which "push"?

How to measure progress?



Framework

① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$

↳ Relaxation of Slows, allows intermediate excess

② Labels/levels $\ell: V \rightarrow \mathbb{R}_{\geq 0}$

↳ Relaxation of distances in residual graph

Push-Relabel Framework

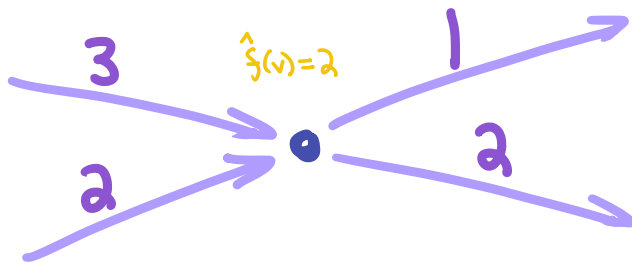
① A "preflow" $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\leftarrow$ Relaxation of flows

- capacity constraints

$$0 \leq f(e) \leq c(e) \quad \forall e \in E$$

- nonnegative net flow $\hat{f}(v) \geq 0 \quad \forall v \neq s$

where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(e) - \sum_{e \in \mathcal{I}^+(v)} f(e)$



net flow at v

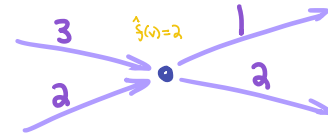
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$\leftarrow$ Relaxation of distances in residual graph

Push-Relabel Framework

Residual graph $G_f = (V, E_f)$

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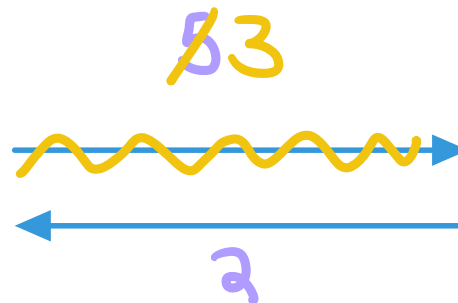
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Relaxation of distances in residual graph

just like for flows:

$$c_f(u,v) = c(u,v) - f(u,v) + f(v,u)$$

residual capacity



② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
 $\uparrow$ Relaxation of distances in residual graph

• Sink has level 0, source has level n

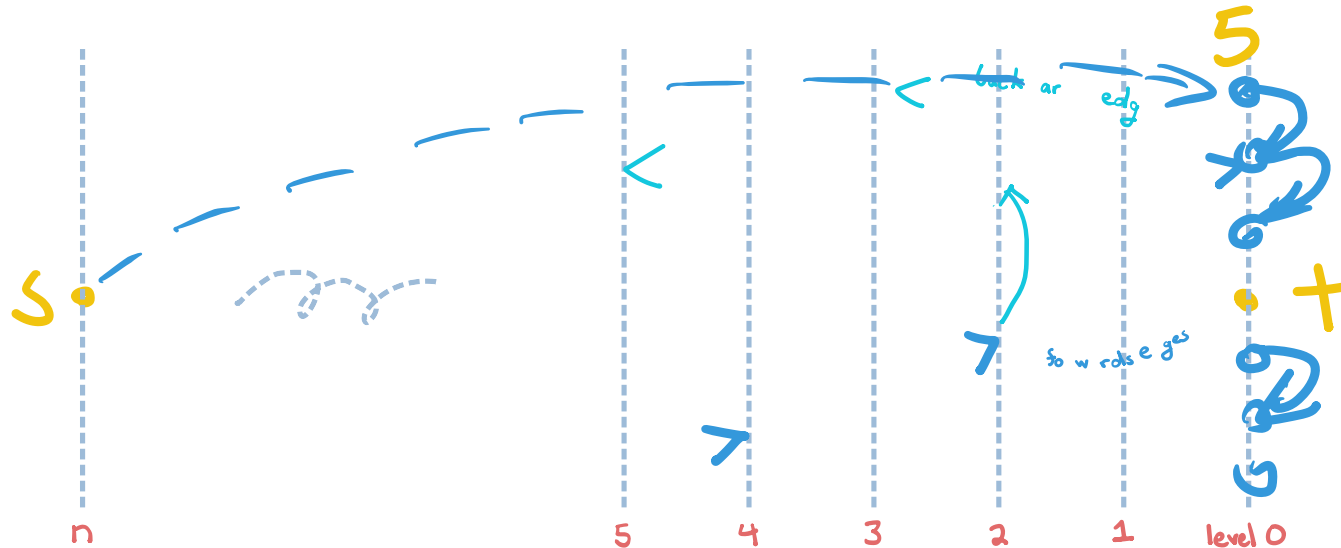
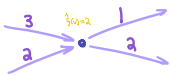
$$l(t) = 0, \quad l(s) = n$$

• edges (in resid. graph) go down ≤ 1 level

$$l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$$

Push-Relabel Framework

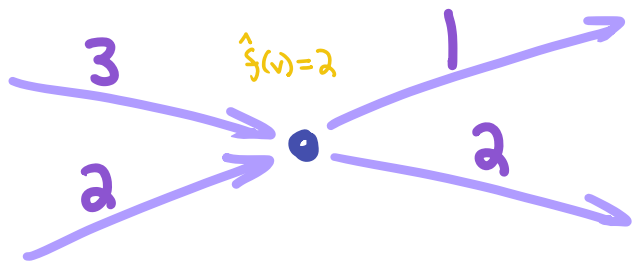
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 net flow at v



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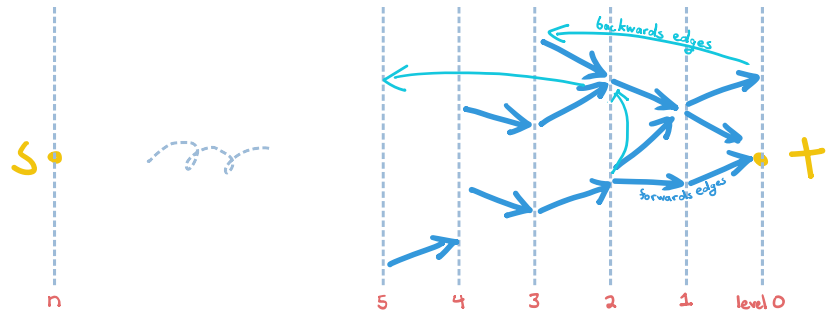
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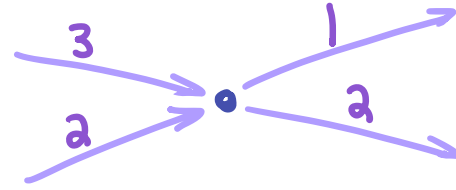
Initially: saturate all source edges, set all non-terminal levels to 0

$$f(s,v) = c(s,v) \quad \forall (s,v) \in \mathcal{I}^+(s)$$

$$\text{level}(s) = n, \quad \text{level}(v) = 0 \quad \forall v \neq t$$

call a vertex u "active" if

$$\hat{f}(u) > 0$$



high-level goal:

push flow out of u "towards t "

Push-Relabel Operations

① Push flow along forward edge

② Relabel vertex to higher level

(while following the rules)

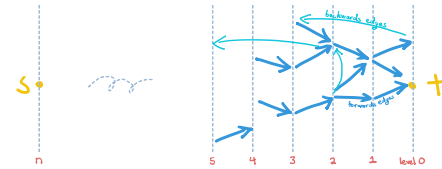
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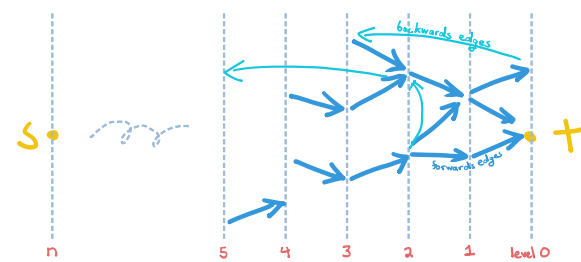
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① Push flow along forward edge

$e = (u,v) \in E_f$ w/ $l(u) = l(v) + 1$

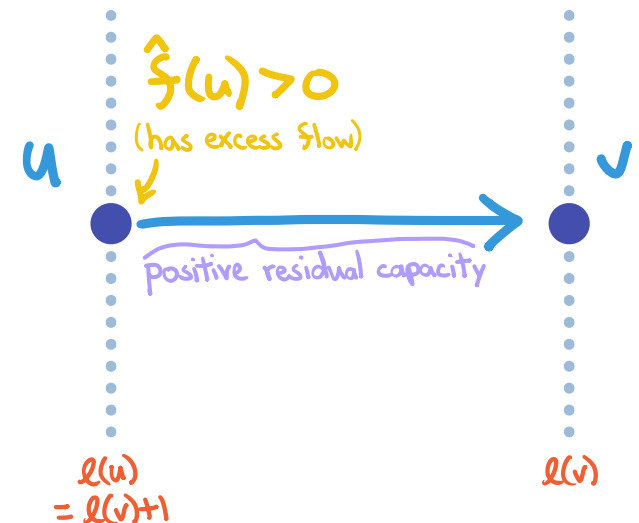
u active

(usually)

push as much flow as possible:

$$\min \{ c_f(e), \hat{f}(u) \}$$

(residual capacity) (excess flow)



② Relabel vertex to higher level

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

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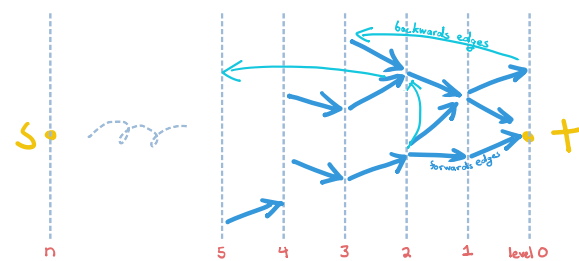
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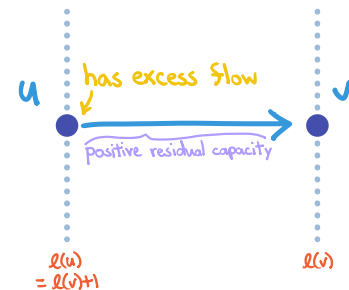
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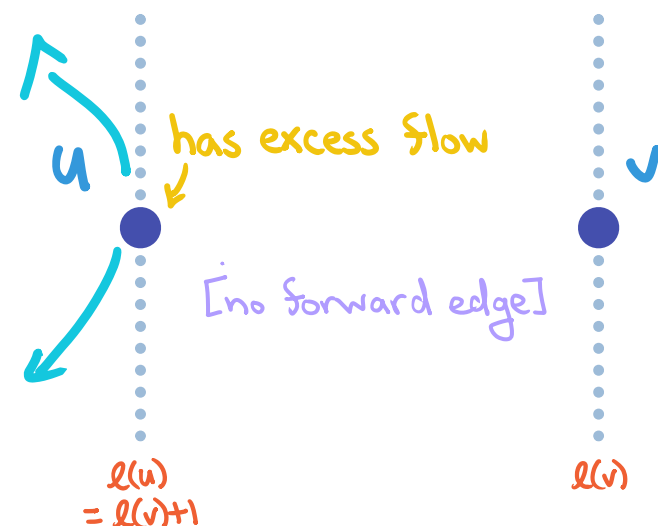
① Push flow along forward edge



② Relabel vertex to higher level

u active, no forward edge

safe to set $l(u) = l(u) + 1$



Algorithm:

① while there is active $v \in V - \{s, t\}$

 @ try to push flow out of v

 ⑥ otherwise relabel v

WTF:

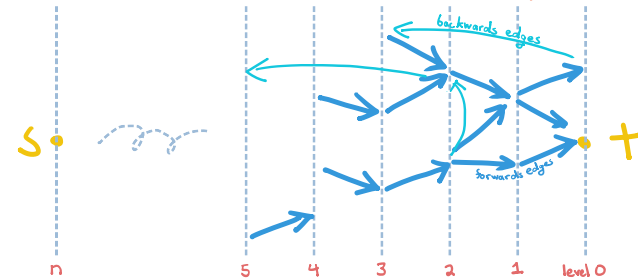
- flow? (let alone max flow?)
- running time?
- does it even terminate?

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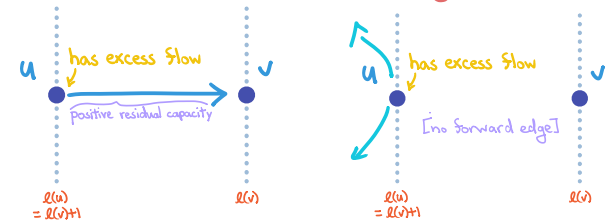
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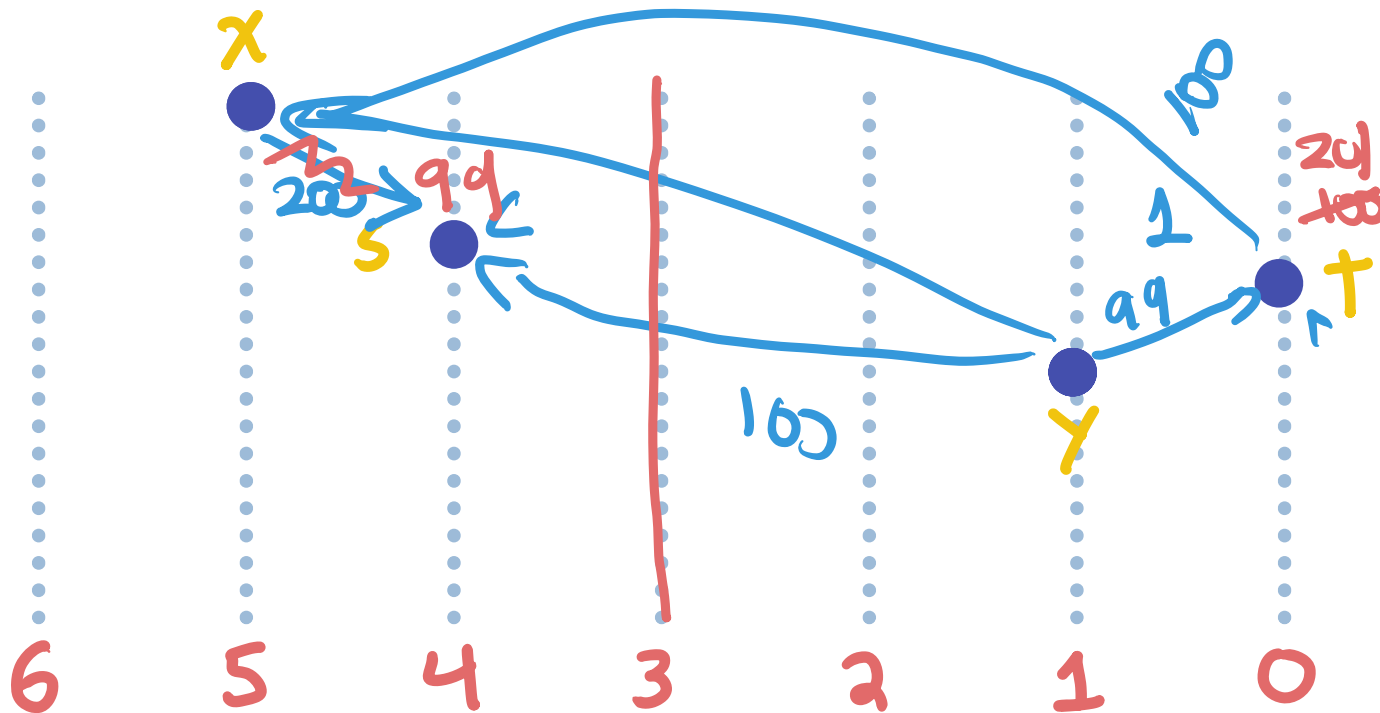
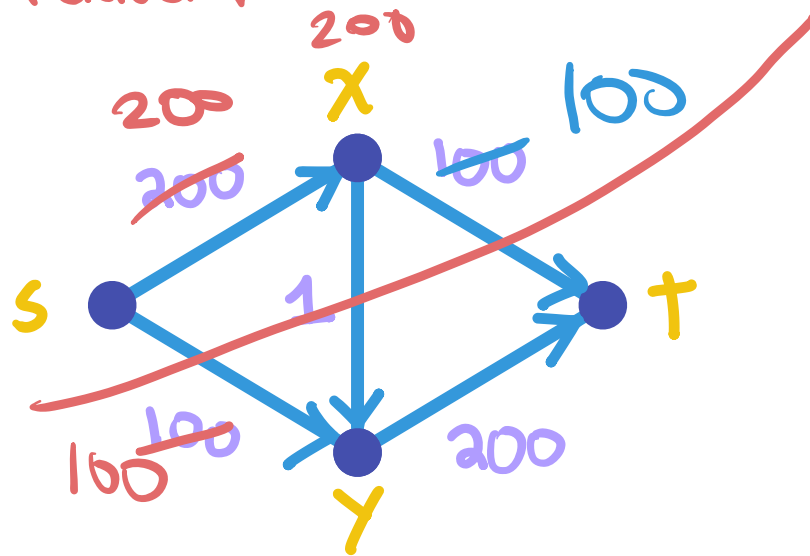
② Relabel vertex to higher level



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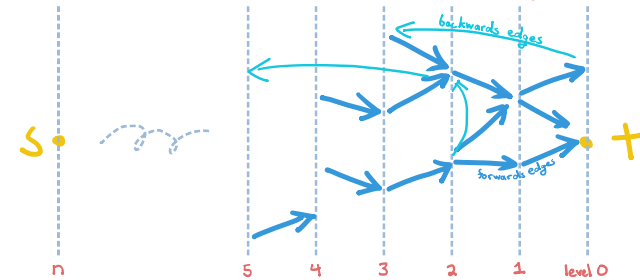
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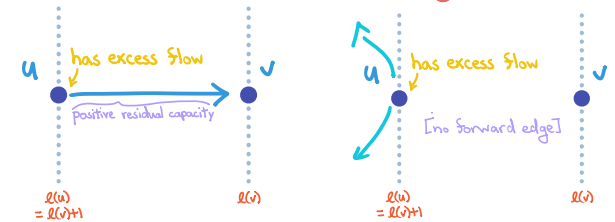
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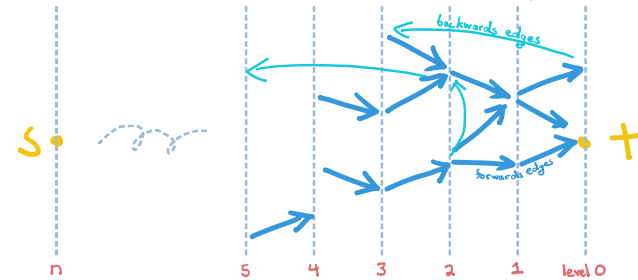
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where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}(v)} f(u, v) - \sum_{e \in \mathcal{O}(v)} f(v, u)$

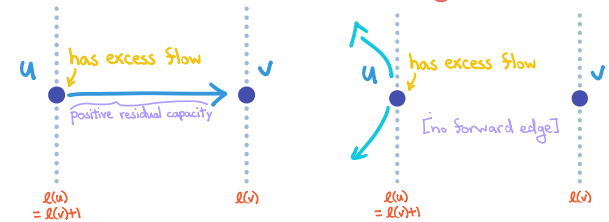
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

- sink has level 0, source has level n
 $l(t) = 0, \quad l(s) = n$
- edges (in resid. graph) go down ≤ 1 level
 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Understanding "preflows"

Lemma let f be a preflow.
then f is a path packing w/
 $\hat{f}(v)$ (s,v) -paths for all v

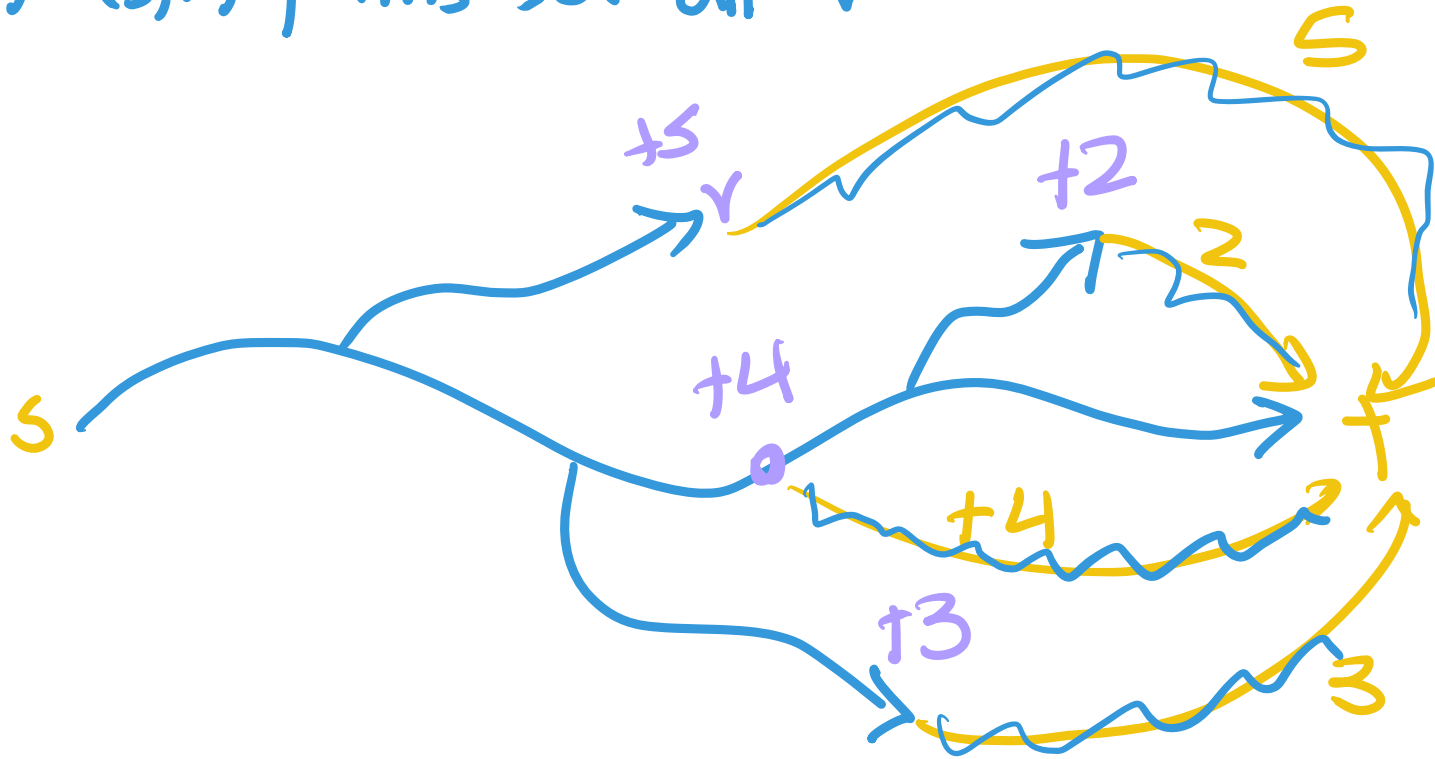
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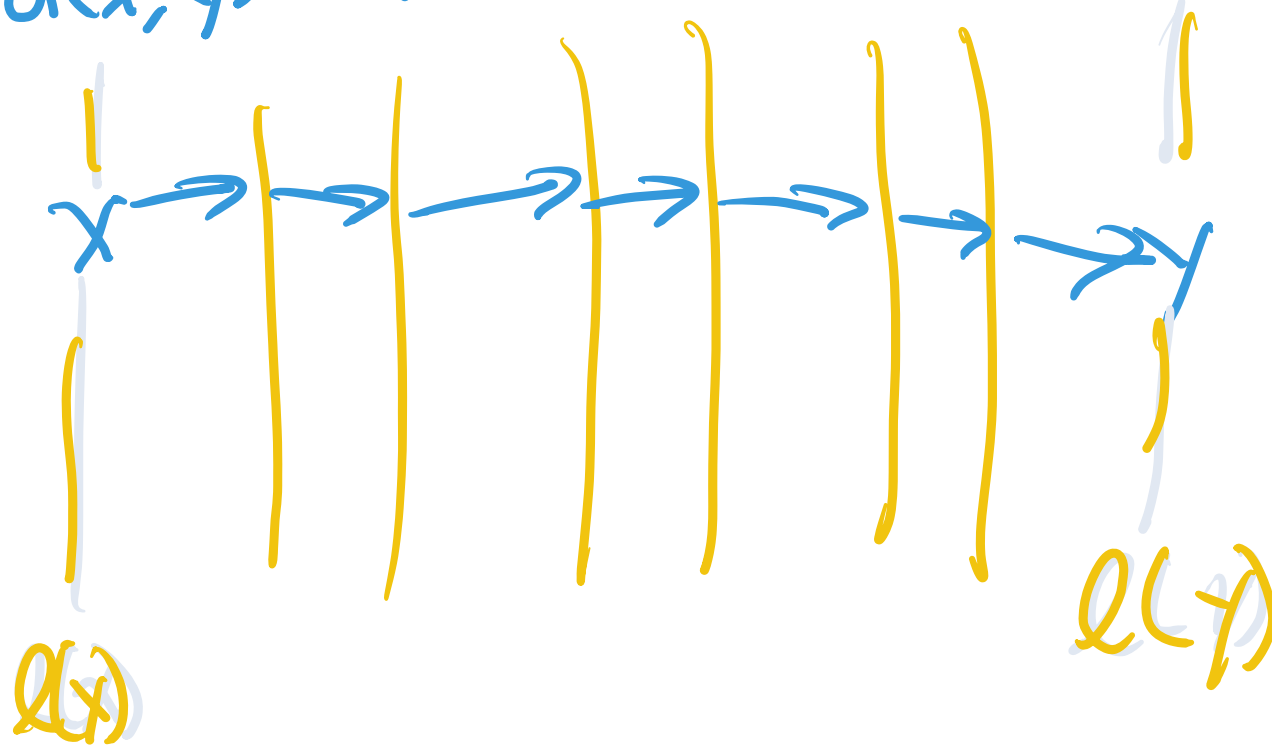
Understanding "labels"

Lemma $\forall x, y \in V,$

$$l(x) - l(y) \leq d(x, y)$$

where $d(x, y)$ = distance in resid. graph

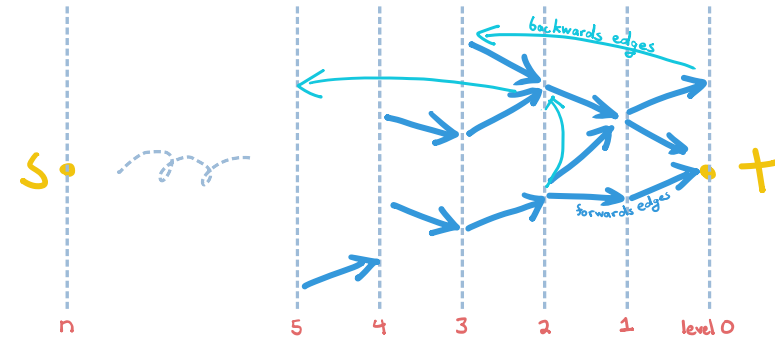
$$d(x, y) = 7$$



② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

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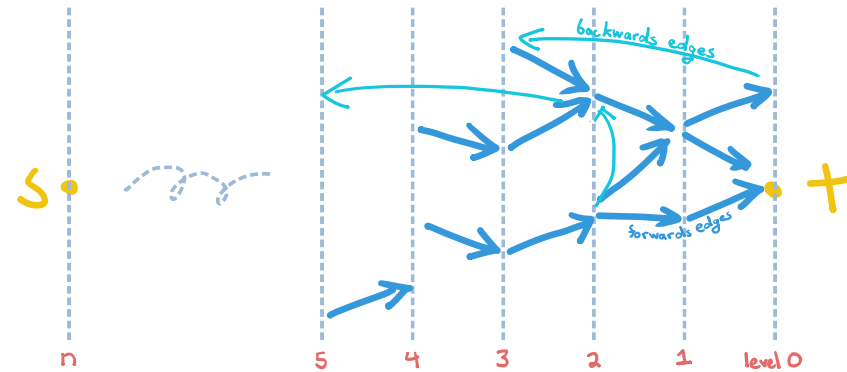
where $d(x, y)$ = distance in resid. graph

$$\Rightarrow d(s, t) \geq l(s) - l(t) = n$$

Corollary: No (s, t) -path in residual graph!

② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of distances in residual graph

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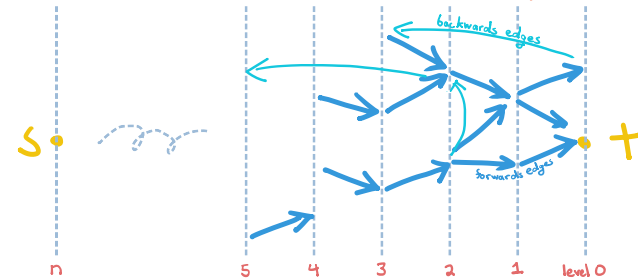
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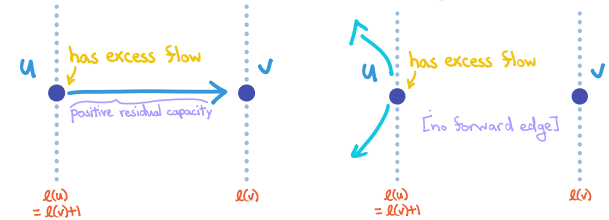
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① Push flow along forward edge

② Relabel vertex to higher level



Lemma if v active, then

$$l(v) < 2n$$

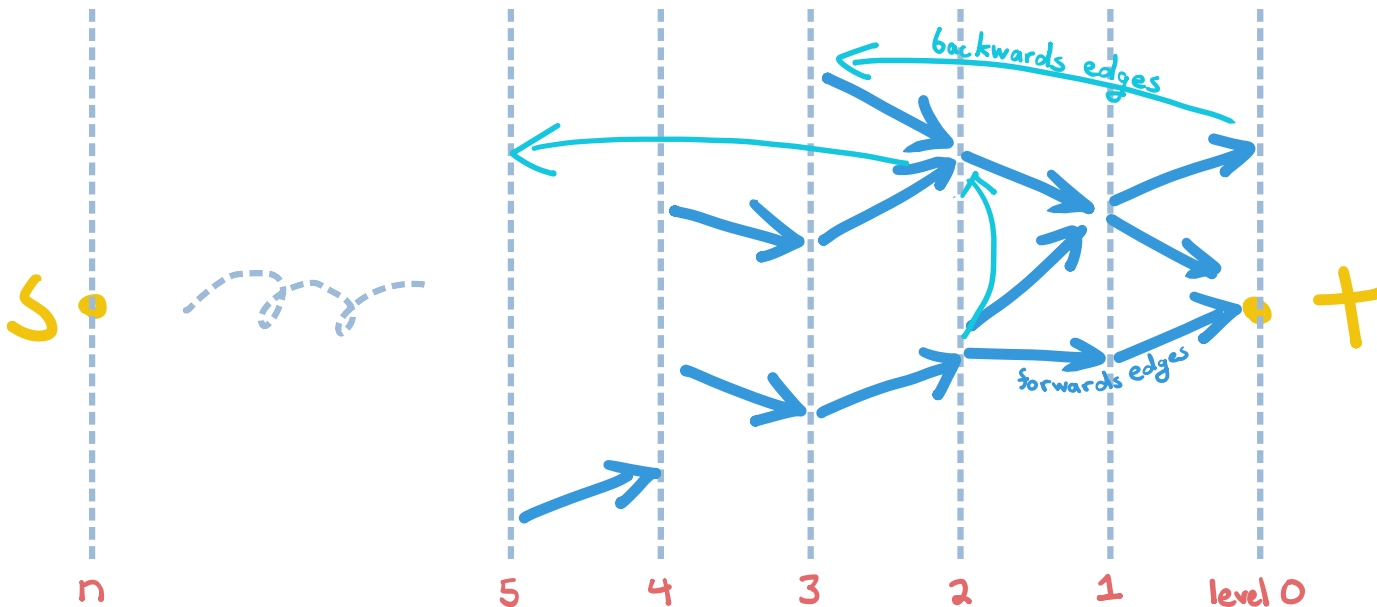
path
decomp $\Rightarrow v \rightsquigarrow s$ path
in residual

Lemma f preflows, $l: V \rightarrow \mathbb{Z}_{\geq 0}$ levels
(no active vertices)

if f is a flow, then

① f is a max flow.

② there is empty layer $i \in \{1, \dots, n-1\}$
s.t. for $U = \{v: l(v) > i\}$,
 $\partial^+(U)$ is a min (s, t) -cut

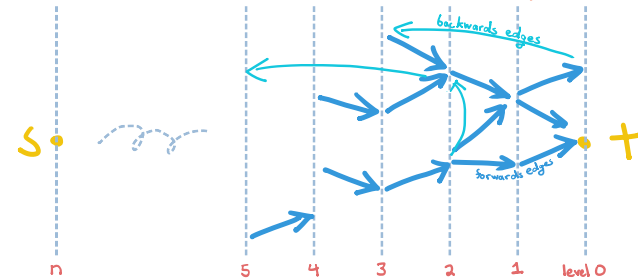


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Relaxation of flows

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where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \partial^-(v)} f(e) - \sum_{e \in \partial^+(v)} f(e)$

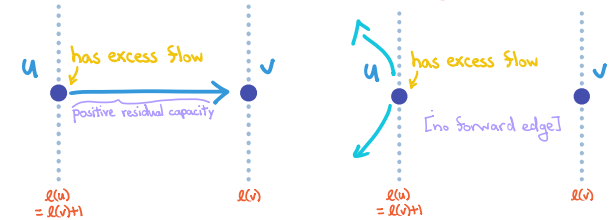
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(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

when no v 's are active,
 f is a max flow.

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

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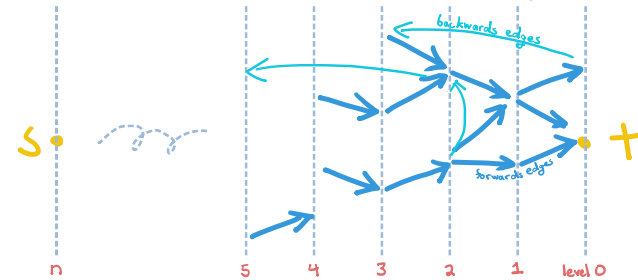
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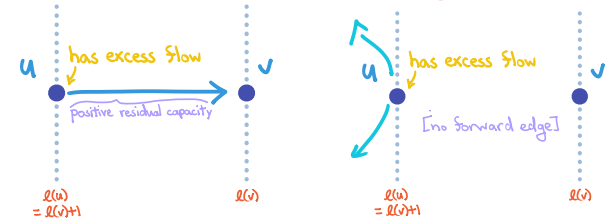
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(Algorithm)

① while there is active $v \in V - \{s, t\}$

 Ⓐ try to push flow out of v

 Ⓑ otherwise relabel v

Now we know why algo is correct.

next: running time

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
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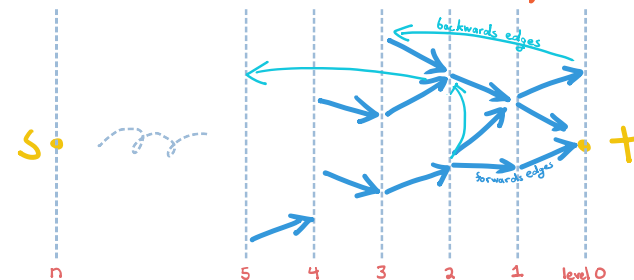
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② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$

$\sim$ Relaxation of distances in residual graph

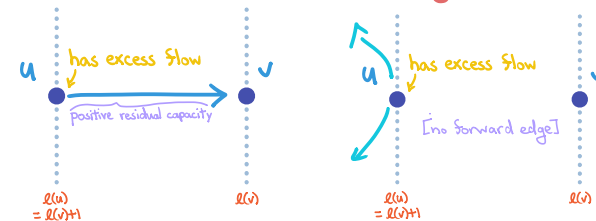
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① Push flow along forward edge

② Relabel vertex to higher level



(Algorithm)

① while there is active $v \in V - \{s, t\}$

② try to push flow out of v

③ otherwise relabel v

Two kinds of pushes:

① Saturating push

uses all capacity of edge

② Non-saturating push

instead pushes out all excess flow from v

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

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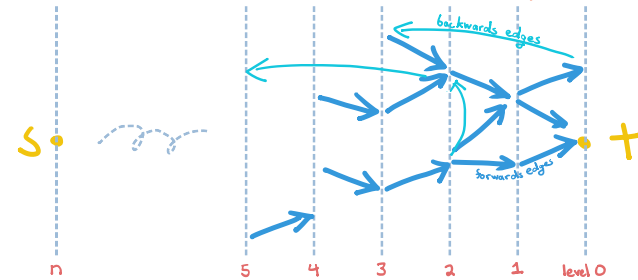
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 $\sim$ Relaxation of distances in residual graph

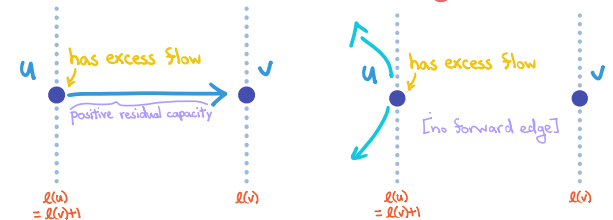
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(Algorithm)

- ① while there is active $v \in V - \{s, t\}$
 - Ⓐ try to push flow out of v
 - Ⓑ otherwise relabel v

we will bound:

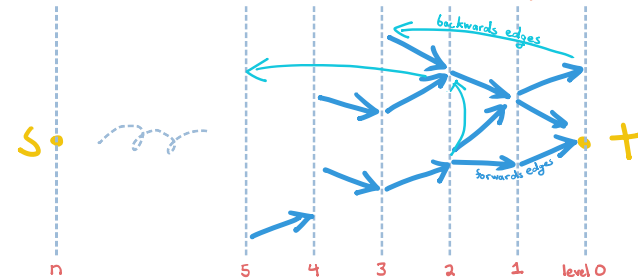
- ① # relabelings
- ② # saturating push
- ③ # non-saturating pushes

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

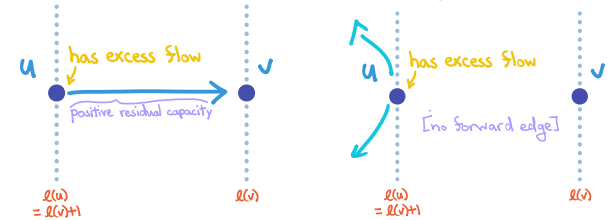
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- ① Push flow along forward edge
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Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

$$\min = 0$$

$$\max = 2n - 1$$

only go up

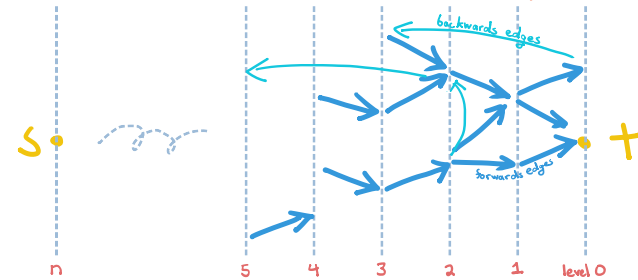
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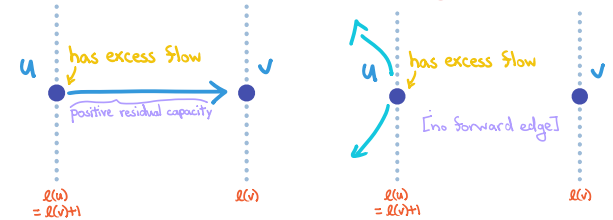
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① Push flow along forward edge

② Relabel vertex to higher level



Lemma

(uses up edge)

(# saturating pushes) $\leq O(mn)$

($O(n)$ per edge)



① while there is active $v \in V - \{s, t\}$

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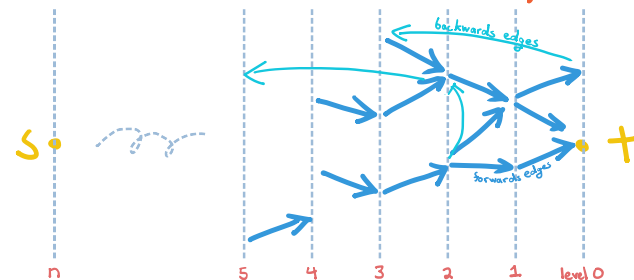
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Relaxation of distances in residual graph

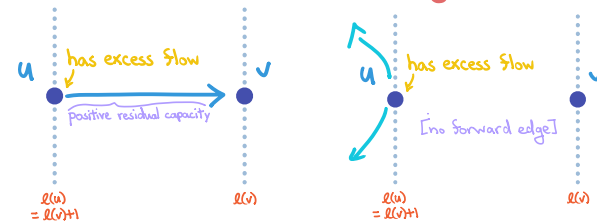
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Lemma

(uses up vertex)

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

Proof ideas?

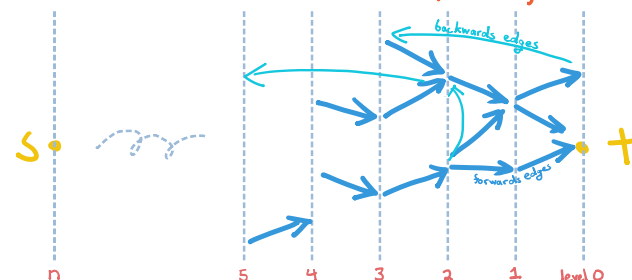
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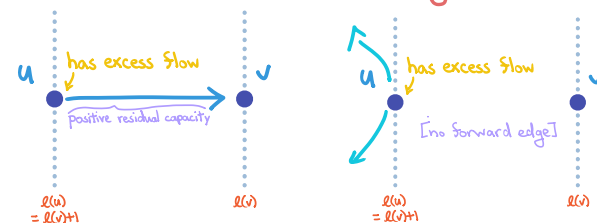
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- ① while there is active $v \in V - \{s, t\}$
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Lemma

(uses up vertex)

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

Proof $\Phi = \sum_{\text{active } v} l(v)$

Φ nonnegative, initially 0

relabel: $\Phi \uparrow 1$

sat push: $\Phi \uparrow \leq O(n)$

Overall: $\Phi \uparrow O(mn^2)$ total

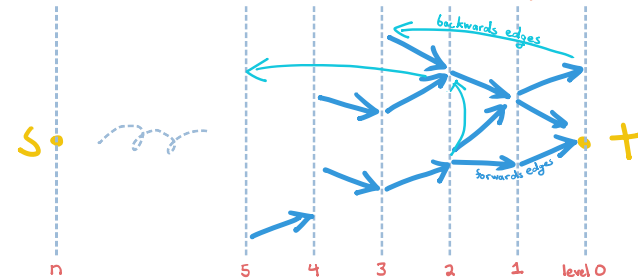
non-sat push: $\Phi \downarrow \geq 1$

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

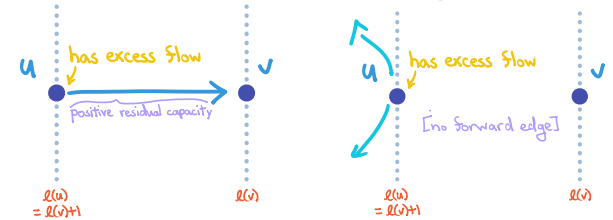
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(Algorithm)

① while there is active $v \in V - \{s, t\}$

ⓐ try to push flow out of v

ⓑ otherwise relabel v

$$(\# \text{ relabels}) \leq O(n^2)$$

$$(\# \text{ saturating pushes}) \leq O(mn)$$

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

+

$$O(mn^2)$$

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
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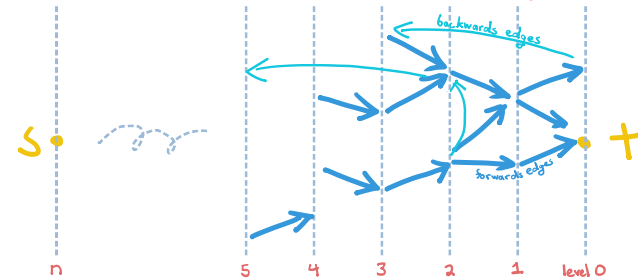
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$\sim$ Relaxation of distances in residual graph

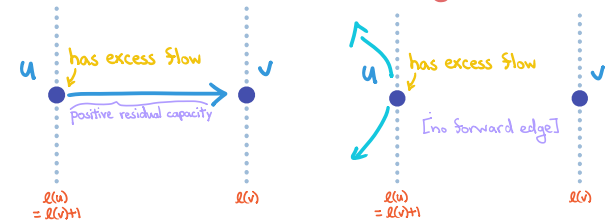
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(Algorithm)

① while there is active $v \in V - \{s, t\}$

 Ⓐ try to push flow out of v

 Ⓑ otherwise relabel v

$O(mn^2)$ time

faster?

Bottleneck was non-saturating pushes

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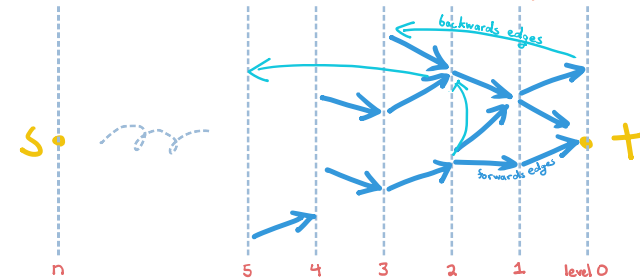
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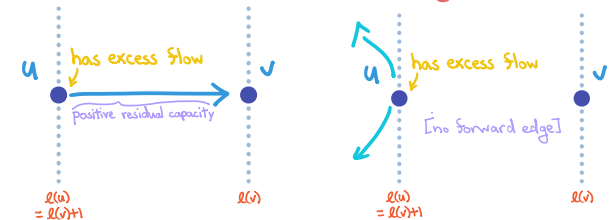
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① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

① while there is active $v \in V - \{s, t\}$

★ let v = active w/ highest level

@ try to push flow out of v

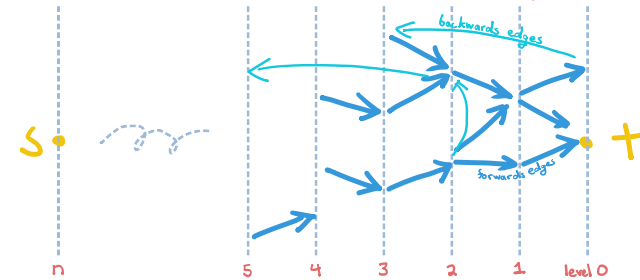
⑥ otherwise relabel v

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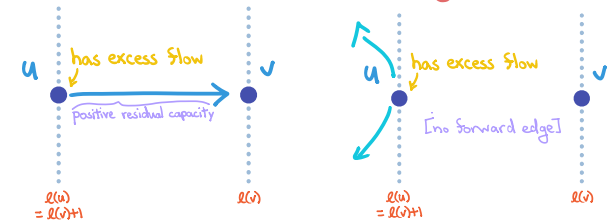
② Labels/levels $l: V \rightarrow \mathbb{R}_{\geq 0}$
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- sink has level 0, source has level n
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 $l(v) \geq l(u) - 1 \quad \forall (u, v) \in E_f$



① Push flow along forward edge

② Relabel vertex to higher level



Top-down push-relabel

- ① while there is active $v \in V - \{s, t\}$
 - ★ let $v = \text{active w/ highest level}$
 - ⓐ try to push flow out of v
 - ⓑ otherwise relabel v

Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn)$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(mn^2)$$

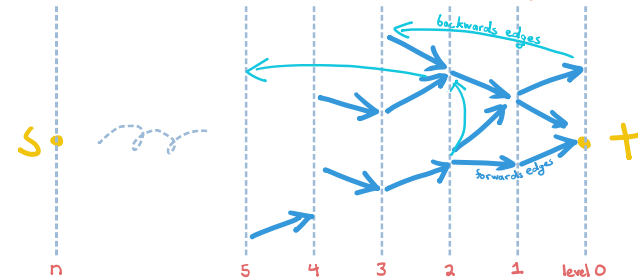
↑ better?

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
 $\sim$ Relaxation of flows

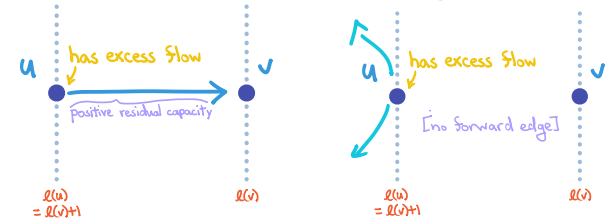
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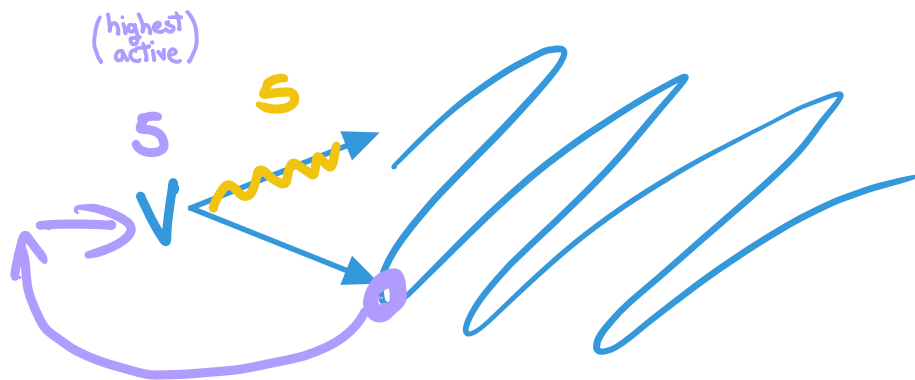


Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$

proof.

suppose non satur. push from v



Top-down push-relabel

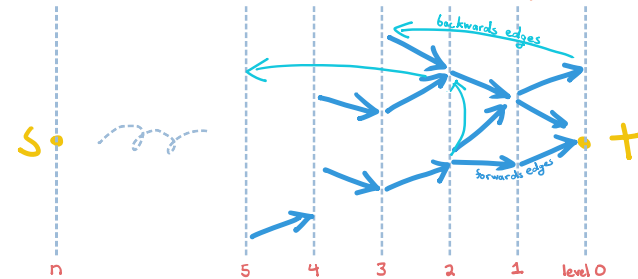
- ① while there is active $v \in V - \{s, t\}$
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① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
Relaxation of flows

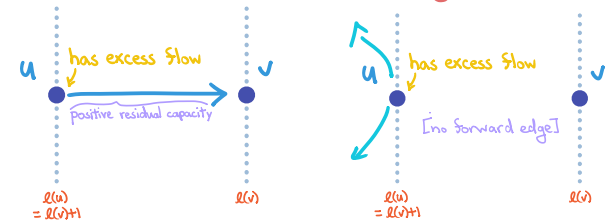
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 where $\hat{f}(v) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{I}^-(v)} f(u, v) - \sum_{e \in \mathcal{I}^+(v)} f(v, u)$

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Lemma

$$(\# \text{ relabels}) \leq O(n^2)$$

Lemma

$$(\# \text{ saturating pushes}) \leq O(mn)$$

Lemma

$$(\# \text{ non-satur. pushes}) \leq O(n^3)$$

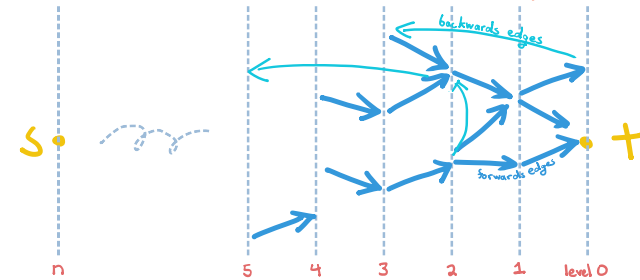
$\Rightarrow O(n^3)$ time overall.

① A preflow $f: E \rightarrow \mathbb{R}_{\geq 0}$
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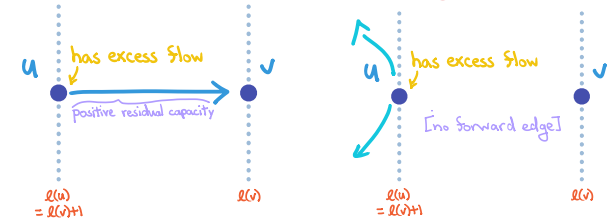
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(better?)

Lemma (# non-satur. pushes) $\leq O(n^2 \sqrt{m})$

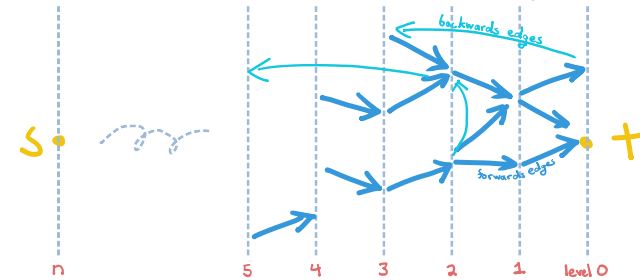
proof. ideas?

① A preflow $\hat{f}: E \rightarrow \mathbb{R}_{\geq 0}$
~ Relaxation of flows

- capacity constraints
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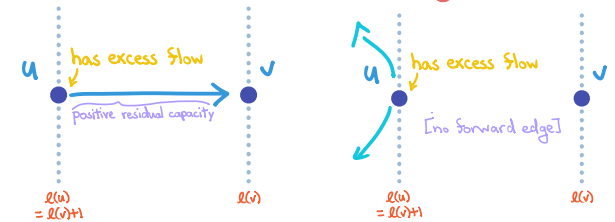
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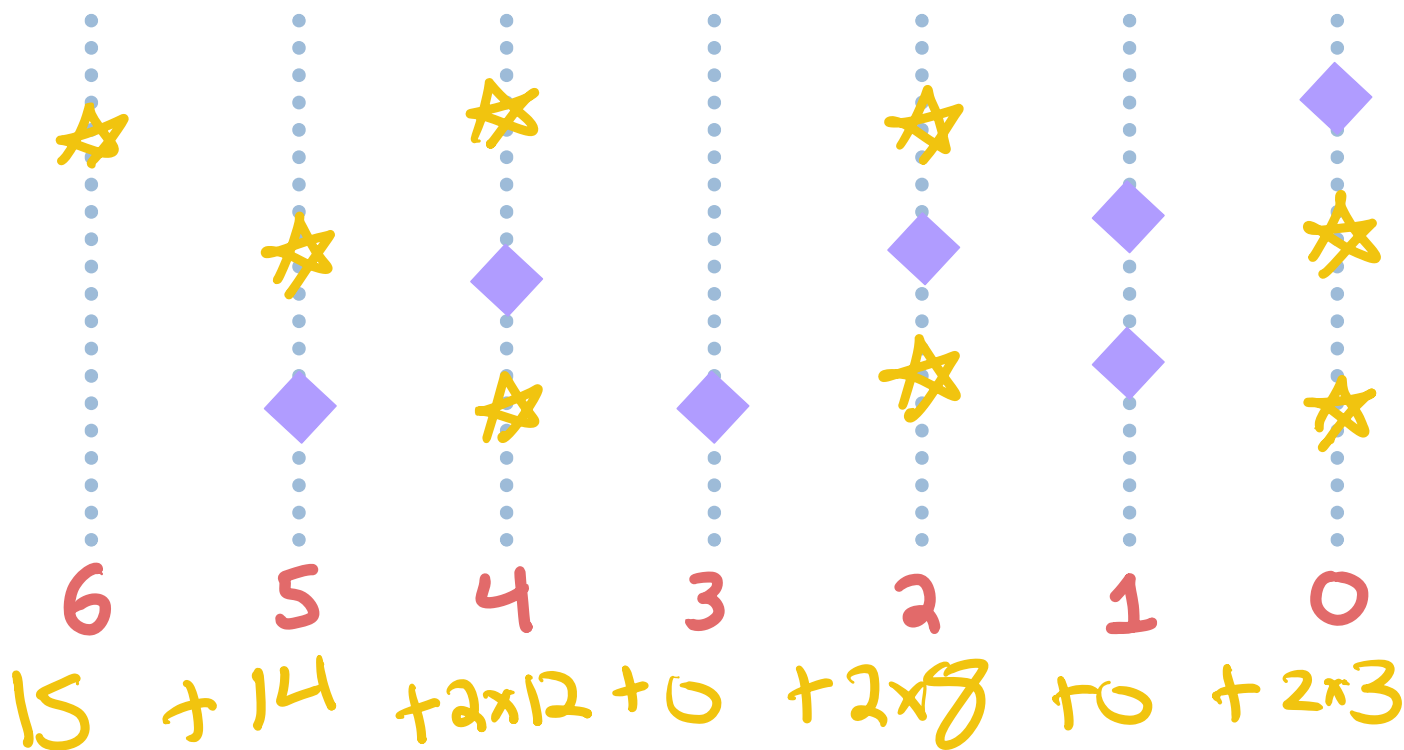
- ① while there is active $v \in V - \{s, t\}$
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Lemma (# non-satur. pushes) $\leq O(n^2 \sqrt{m})$

proof. $\Phi = \sum_{v \text{ active}} |w: l(w) \leq l(v)|$

★ = active

◆ = inactive
15 total

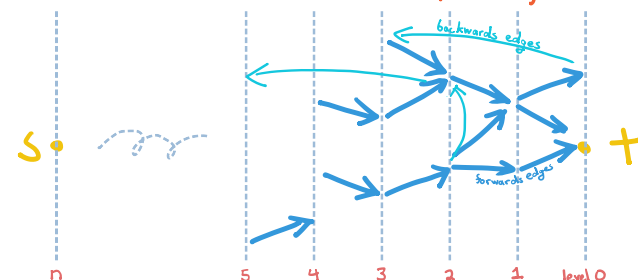


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Relaxation of flows

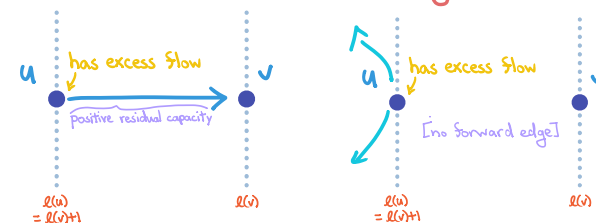
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proof. $\Phi = \sum_{v \text{ active}} |w: \ell(w) \leq \ell(v)|$

relabel, satur. push: $\Delta \Phi \text{ by } \leq n$
 $\Rightarrow$ total increase of $O(mn^2)$

non-satur. push at v : $\Delta \Phi = -|w: \ell(w) = \ell(v)|$

"big" if $|w: \ell(w) = \ell(v)| \geq k$ w/ $k \text{ TBD}$

"small" if $|w: \ell(w) = \ell(v)| \leq k$

claims: $O(mn^2/k)$ big pushes

$O(n^2 k)$ small pushes

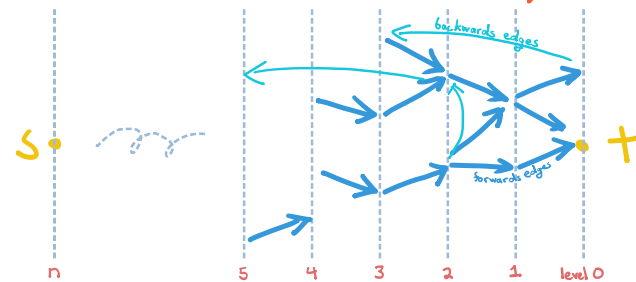
Then set $k = \sqrt{m}$

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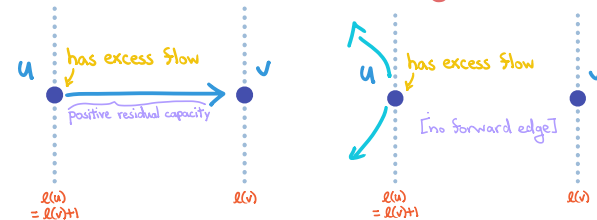
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claim: $O(mn^2/k)$ big pushes

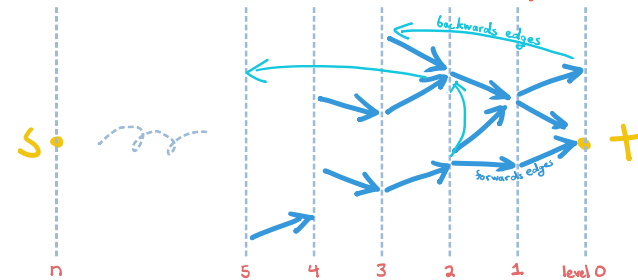
big push decreases Φ by $\geq k$

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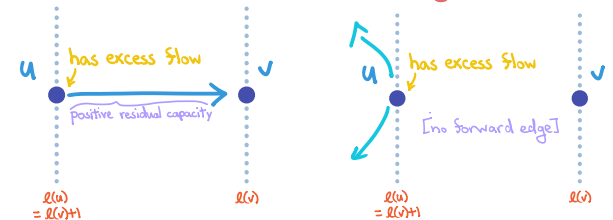
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"small" if $|w: l(w) = l(v)| \leq k$ w/ k TBD

claim: $O(n^2 k)$ small pushes why?

k "small" pushes before active level decreases

n^2 active level decreases

n^2 active level increases

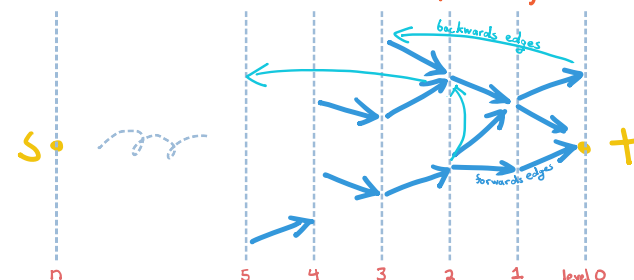
because n^2 relabels

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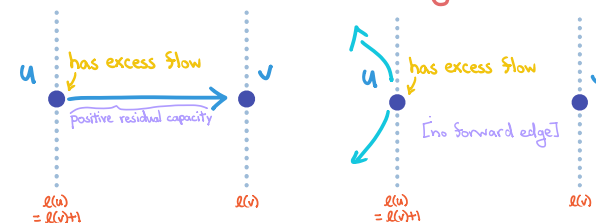
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